k-SHAPE POSET AND BRANCHING OF k-SCHUR FUNCTIONS

THOMAS LAM, LUC LAPOINTE, JENNIFER MORSE, AND MARK SHIMOZONO

Abstract. We give a combinatorial expansion of a Schubert homology class in the affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}_k}$ into Schubert homology classes in $\mathrm{Gr}_{\mathrm{SL}_{k+1}}$. This is achieved by studying the combinatorics of a new class of partitions called *k-shapes*, which interpolates between k-cores and $k+1$-cores. We define a symmetric function for each k-shape, and show that they expand positively in terms of dual k-Schur functions. We obtain an explicit combinatorial description of the expansion of an ungraded k-Schur function into $k+1$-Schur functions. As a corollary, we give a formula for the Schur expansion of an ungraded k-Schur function.

1. Introduction

1.1. **k-Schur functions and branching coefficients.** The theory of k-Schur functions arose from the study of Macdonald polynomials and has since been connected to quantum and affine Schubert calculus, K-theory, and representation theory. The origin of the k-Schur functions is related to Macdonald's positivity conjecture, which asserted that in the expansion

$$H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}(q,t)\, s_\lambda\,, \tag{1}$$

the coefficients $K_{\lambda\mu}(q,t)$, called q,t-Kostka polynomials, belong to $\mathbb{Z}_{\geq 0}[q,t]$. Although the final piece in the proof of this conjecture was made by Haiman [4] using representation theoretic and geometric methods, the long study of this conjecture brought forth many further problems and theories. The study of the q,t-Kostka polynomials remains a matter of great interest.

It was conjectured in [9] that by fixing an integer $k>0$, any Macdonald polynomial indexed by $\lambda \in \mathcal{B}^k$ (the set of partitions such that $\lambda_1 \leq k$) could be decomposed as:

$$H_\mu[X;q,t\,] = \sum_{\lambda\in\mathcal{B}^k} K^{(k)}_{\lambda\mu}(q,t)\, s^{(k)}_\lambda[X;t\,] \quad \text{where} \quad K^{(k)}_{\lambda\mu}(q,t) \in \mathbb{Z}_{\geq 0}[q,t\,]\,, \tag{2}$$

for some symmetric functions $s^{(k)}_\lambda[X;t]$ associated to sets of tableaux called atoms. Conjecturally equivalent characterizations of $s^{(k)}_\lambda[X;t]$ were later given in [10, 8] and the descriptions of [9, 10, 8] are now all generically called (graded) k-Schur functions. A basic property of the k-Schur functions is that

$$s^{(k)}_\lambda[X;t] = s_\lambda \quad \text{for } k \geq |\lambda|\,, \tag{3}$$

and it thus follows that Eq. (2) significantly refines Macdonald's original conjecture since the expansion coefficient $K^{(k)}_{\lambda\mu}(q,t)$ reduces to $K_{\lambda\mu}(q,t)$ for large k.

1991 *Mathematics Subject Classification.* Primary 05E05, 05E10; Secondary 14N35, 17B65.

Furthermore, it was conjectured that the k-Schur functions satisfy a highly structured filtration, which is our primary focus here. To be precise:

Conjecture 1. *For $k' > k$ and partitions $\mu \in \mathcal{B}^k$ and $\lambda \in \mathcal{B}^{k'}$, there are polynomials $\tilde{b}_{\mu\lambda}^{(k\to k')}(t) \in \mathbb{Z}_{\geq 0}[t]$ such that*

$$s_\mu^{(k)}[X;t] = \sum_{\lambda \in \mathcal{B}^{k'}} \tilde{b}_{\mu\lambda}^{(k\to k')}(t)\, s_\lambda^{(k')}[X;t]. \tag{4}$$

In particular, the Schur function expansion of a k-Schur function is obtained from (3) and (4) by letting $k' \to \infty$. The remarkable property described in Conjecture 1 provides a step-by-step approach to understanding k-Schur functions since the polynomials $\tilde{b}_{\mu\lambda}^{(k\to k')}(t)$ can be expressed positively in terms of the *branching polynomials*

$$\tilde{b}_{\mu\lambda}^{(k)}(t) := \tilde{b}_{\mu\lambda}^{(k-1\to k)}(t)\,,$$

via iteration (tables of branching polynomials are given in Appendix A).

It has also come to light that ungraded k-Schur functions (the case when $t=1$) are intimately tied to problems in combinatorics, geometry, and representation theory beyond the theory of Macdonald polynomials. Thus, understanding the *branching coefficients*,

$$\tilde{b}_{\mu\lambda}^{(k)} := \tilde{b}_{\mu\lambda}^{(k)}(1)$$

gives a step-by-step approach to problems in areas such as affine Schubert calculus and K-theory (for example, see §§1.4).

Our work here gives a combinatorial description for the branching coefficients, proving Conjecture 1 when $t=1$. We use the ungraded k-Schur functions $s_\lambda^{(k)}[X]$ defined in [12], which coincide with those defined in [8] terms of strong k-tableaux. Moreover, we conjecture a formula for the branching polynomials in general. The combinatorics behind these formulas involves a certain *k-shape poset*.

1.2. **k-shape poset.** A key development in our work is the introduction of a new family of partitions called k-shapes and a poset on these partitions (see §2 for full details and examples). Our formula for the branching coefficients is given in terms of path enumeration in the k-shape poset.

For any partition λ identified by its Ferrers diagram, we define its *k-boundary* $\partial\lambda$ to be the cells of λ with hook-length no greater than k. $\partial\lambda$ is a skew shape, to which we associate compositions $\mathrm{rs}(\lambda)$ and $\mathrm{cs}(\lambda)$, where $\mathrm{rs}(\lambda)_i$ (resp. $\mathrm{cs}(\lambda)_i$) is the number of cells in the i-th row (resp. column) of $\partial\lambda$. A partition λ is said to be a *k-shape* if both $\mathrm{rs}(\lambda)$ and $\mathrm{cs}(\lambda)$ are partitions. The rank of k-shape λ is defined to be $|\partial\lambda|$, the number of cells in its k-boundary. Π^k denotes the set of all k-shapes.

We introduce a poset structure on Π^k where the partial order is generated by distinguished downward relations in the poset called *moves* (Definition 19). The set of k-shapes contains the set $\mathcal{C}^k$ of all k-cores (partitions with no cells of hook-length k) *and* the set $\mathcal{C}^{k+1}$ of $k+1$-cores. Moreover, the maximal elements of Π^k are given by $\mathcal{C}^{k+1}$ and the minimal elements by $\mathcal{C}^k$. In Definition 36 we give a charge statistic on moves from which we obtain an equivalence relation on paths (sequences of moves) in Π^k; roughly speaking, two paths are equivalent if they are related by a sequence of charge-preserving diamonds (see Eqs. (42)-(44)). Charge is thus constant on equivalence classes of paths.

For $\lambda, \mu \in \Pi^k$, $\mathcal{P}^k(\lambda,\mu)$ is the set of paths in Π^k from λ to μ and $\overline{\mathcal{P}}^k(\lambda,\mu)$ is the set of equivalence classes in $\mathcal{P}^k(\lambda,\mu)$. Our main result is that the branching coefficients enumerate these equivalence classes. To be precise, for $\lambda \in \mathcal{C}^{k+1}$ and $\mu \in \mathcal{C}^k$, set

$$b^{(k)}_{\mu\lambda}(t) := \tilde{b}^{(k)}_{\mathrm{rs}(\mu)\mathrm{rs}(\lambda)}(t) \tag{5}$$

$$b^{(k)}_{\mu\lambda} := \tilde{b}^{(k)}_{\mathrm{rs}(\mu)\mathrm{rs}(\lambda)} \tag{6}$$

so that

$$s^{(k-1)}_{\mu}[X] = \sum_{\lambda\in\mathcal{C}^{k+1}} b^{(k)}_{\mu\lambda} s^{(k)}_{\lambda}[X]. \tag{7}$$

Hereafter, we will label k-Schur functions by cores rather than k-bounded partitions using the bijection between $\mathcal{C}^{k+1}$ and $\mathcal{B}^k$ given by the map rs.

Theorem 2. *For all $\lambda \in \mathcal{C}^{k+1}$ and $\mu \in \mathcal{C}^k$,*

$$b^{(k)}_{\mu\lambda} = |\overline{\mathcal{P}}^k(\lambda,\mu)|. \tag{8}$$

We conjecture that the charge statistic on paths gives the branching polynomials.

Conjecture 3. *For all $\lambda \in \mathcal{C}^{k+1}$ and $\mu \in \mathcal{C}^k$,*

$$b^{(k)}_{\mu\lambda}(t) = \sum_{[\mathbf{p}]\in\overline{\mathcal{P}}^k(\lambda,\mu)} t^{\mathrm{charge}(\mathbf{p})}. \tag{9}$$

1.3. **k-shape functions.** The proof of Theorem 2 relies on the introduction of a new family of symmetric functions indexed by k-shapes. These functions generalize the dual (affine/weak) k-Schur functions studied in [13, 5, 8].

The images of the dual k-Schur functions $\{\mathrm{Weak}^{(k)}_{\lambda}[X]\}_{\lambda\in\mathcal{C}^{k+1}}$ form a basis for the quotient

$$\Lambda/I_k \quad \text{where} \quad I_k = \langle m_\lambda : \lambda_1 > k\rangle \tag{10}$$

of the space Λ of symmetric functions over $\mathbb{Z}$, while the ungraded k-Schur functions $\{s^{(k)}_{\lambda}[X]\}_{\lambda\in\mathcal{C}^{k+1}}$ form a basis for the subring $\Lambda^{(k)} = \mathbb{Z}[h_1,\dots,h_k]$ of Λ. The Hall inner product $\langle\cdot\,,\cdot\rangle : \Lambda\times\Lambda\to\mathbb{Z}$ is defined by $\langle m_\lambda\,, h_\mu\rangle = \delta_{\lambda\mu}$. For each k there is an induced perfect pairing $\langle\cdot\,,\cdot\rangle_k : \Lambda/I_k \times \Lambda^{(k)} \to \mathbb{Z}$, and it was shown in [13] that

$$\langle \mathrm{Weak}^{(k)}_{\lambda}[X]\,, s^{(k)}_{\mu}[X]\rangle_k = \delta_{\lambda\mu} \tag{11}$$

Moreover, it was shown in [6] that $\{\mathrm{Weak}^{(k)}_{\lambda}[X]\}_{\lambda\in\mathcal{C}^{k+1}}$ represents Schubert classes in the cohomology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$ of SL_{k+1}.

The original characterization of $\mathrm{Weak}^{(k)}_{\lambda}[X]$ was given in [13] using k-tableaux. A k-tableau encodes a sequence of $k+1$-cores

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)} = \lambda\,, \tag{12}$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ are certain horizontal strips. The weight of a k-tableau T is

$$\mathrm{wt}(T) = (a_1, a_2, \dots, a_N) \quad \text{where} \quad a_i = |\partial\lambda^{(i)}| - |\partial\lambda^{(i-1)}|\,. \tag{13}$$

For λ a $k+1$-core, the dual k-Schur function is defined as the weight generating function

$$\mathrm{Weak}^{(k)}_{\lambda}[X] = \sum_{T\in\mathrm{WTab}^k_\lambda} x^{\mathrm{wt}(T)}\,, \tag{14}$$

where WTab^k_λ is the set of k-tableaux of shape λ.

Here we consider *k-shape tableaux*. These are defined similarly, but now we allow the shapes in (12) to be k-shapes and $\lambda^{(i)}/\lambda^{(i-1)}$ are certain *reverse-maximal* strips (defined in §4). The weight is again defined by (13) and for each k-shape λ, we then define the *cohomology k-shape function* $\mathfrak{S}^{(k)}_\lambda$ to be the weight generating function

$$\mathfrak{S}^{(k)}_\lambda[X] = \sum_{T \in \mathrm{Tab}^k_\lambda} x^{\mathrm{wt}(T)}\,, \tag{15}$$

where Tab^k_λ denotes the set of reverse-maximal k-shape tableaux of shape λ.

We show the k-shape functions are symmetric and that when λ is a $k+1$-core,

$$\mathrm{Weak}^{(k)}_\lambda[X] = \mathfrak{S}^{(k)}_\lambda[X] \mod I_{k-1}\,, \tag{16}$$

(see Proposition 73). We give a combinatorial expansion of any k-shape function in terms of dual $(k-1)$-Schur functions.

Theorem 4. *For $\lambda \in \Pi^k$, the cohomology k-shape function $\mathfrak{S}^{(k)}_\lambda[X]$ is a symmetric function with the decomposition*

$$\mathfrak{S}^{(k)}_\lambda[X] = \sum_{\mu \in \mathcal{C}^k} |\overline{\mathcal{P}}^k(\lambda,\mu)|\, \mathrm{Weak}^{(k-1)}_\mu[X]\,. \tag{17}$$

It is from this theorem that we deduce Theorem 2. Letting $\lambda \in \mathcal{C}^{k+1}$ and $\mu \in \mathcal{C}^k$, we have

$$\begin{aligned} b^{(k)}_{\mu\lambda} &= \langle \mathrm{Weak}^{(k)}_\lambda[X]\,,\, s^{(k-1)}_\mu[X]\rangle_k \\ &= \langle \mathrm{Weak}^{(k)}_\lambda[X]\,,\, s^{(k-1)}_\mu[X]\rangle_{k-1} \\ &= \langle \mathfrak{S}^{(k)}_\lambda[X]\,,\, s^{(k-1)}_\mu[X]\rangle_{k-1} \\ &= \overline{\mathcal{P}}^k(\lambda,\mu) \end{aligned}$$

using (7), (11) for $k-1$, (16), and Theorem 4.

A (homology) k-shape function can also be defined for each k-shape μ by

$$\mathfrak{s}^{(k)}_\mu[X;t] = \sum_{\lambda \in \mathcal{C}^{k+1}} \sum_{[\mathbf{p}] \in \overline{\mathcal{P}}^k(\lambda,\mu)} t^{\mathrm{charge}(\mathbf{p})}\, s^{(k)}_\lambda[X;t]\,, \tag{18}$$

and its ungraded version is $\mathfrak{s}^{(k)}_\mu[X] := \mathfrak{s}^{(k)}_\mu[X;1]$. We trivially have from this definition that $\mathfrak{s}^{(k)}_\mu[X] = s^{(k)}_\mu[X]$ when $\mu \in \mathcal{C}^{k+1}$. Further, from (18) at $t=1$, Theorem 2 and (7), we have that

$$\mathfrak{s}^{(k)}_\mu[X] = s^{(k-1)}_\mu[X] \qquad \text{for } \mu \in \mathcal{C}^k. \tag{19}$$

The Pieri rule for ungraded homology k-shape functions is given by

Theorem 5. *For $\lambda \in \Pi^k$ and $r \le k-1$, one has*

$$h_r[X]\,\mathfrak{s}^{(k)}_\lambda[X] = \sum_{\nu \in \Pi^k} \mathfrak{s}^{(k)}_\nu[X]$$

where the sum is over maximal strips ν/λ of rank r.

When λ is a k-core, Theorem 5 implies the Pieri rule for $(k-1)$-Schur functions proven in [12].

Here we have introduced the cohomology k-shape functions as the generating function of tableaux that generalize k-tableaux (those defining the dual k-Schur functions). There is another family of "strong k-tableaux" whose generating functions are k-Schur functions [8]. The generalization of this family to give a direct characterization of homology k-shape functions remains an open problem (see §§1.5 for further details).

Theorems 4 and 5 are proved using an explicit bijection (Theorem 75):

$$\begin{aligned} \mathrm{Tab}_\lambda^k &\longrightarrow \bigsqcup_{\mu \in \mathcal{C}^k} \mathrm{WTab}_\mu^k \times \overline{\mathcal{P}}^k(\lambda,\mu) \\ T &\longmapsto (U,[\mathbf{p}]) \end{aligned} \tag{20}$$

such that $\mathrm{wt}(T) = \mathrm{wt}(U)$. The bulk of this article is in establishing this bijection, which requires many intricate details. See §§1.8 for pointers to the highlights of our development.

1.4. **Geometric meaning of branching coefficients.** It is proven in [6] that the ungraded k-Schur functions are Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$ of SL_{k+1}. The ind-scheme $\mathrm{Gr}_{SL_{k+1}}$ is an affine Kac-Moody homogeneous space and the homology ring $H_*(\mathrm{Gr}_{\mathrm{SL}_{k+1}})$ has a basis of fundamental homology classes $[X_\lambda]_*$ of Schubert varieties $X_\lambda \subset \mathrm{Gr}_{\mathrm{SL}_{k+1}}$, and $H^*(\mathrm{Gr}_{\mathrm{SL}_{k+1}})$ has the dual basis $[X_\lambda]^*$, where λ runs through the set of $k+1$-cores.

There is a weak homotopy equivalence between $\mathrm{Gr}_{\mathrm{SL}_{k+1}}$ and the topological group ΩSU_{k+1} of based loops $(S^1,1) \to (SU_{k+1},\mathrm{id})$ into SU_{k+1}. This induces isomorphisms of dual Hopf algebras $H_*(\Omega SU_{k+1}) \cong H_*(\mathrm{Gr}_{\mathrm{SL}_{k+1}})$ and $H^*(\Omega SU_{k+1}) \cong H^*(\mathrm{Gr}_{SL_{k+1}})$. The Pontryagin product in $H_*(\Omega SU_{k+1})$ is induced by the product in the group ΩSU_{k+1}.

Using Peterson's characterization of the Schubert basis of $H_*(\mathrm{Gr}_{SL_{k+1}})$ and the definition of [12] for $s_\lambda^{(k)}[X]$, it is shown in [6] that there is a Hopf algebra isomorphism

$$\begin{aligned} H_*(\mathrm{Gr}_{\mathrm{SL}_{k+1}}) \cong H_*(\Omega SU_{k+1}) &\xrightarrow{j^{(k)}} \mathbb{Z}[h_1,h_2,\dots,h_k] \subset \Lambda \\ [X_\lambda]_* &\longmapsto s_\lambda^{(k)}[X] \end{aligned} \tag{21}$$

mapping homology Schubert classes to k-Schur functions.

Let $i^{(k)} : \Omega SU_k \to \Omega SU_{k+1}$ be the inclusion map and $i_*^{(k)} : H_*(\Omega SU_k) \to H_*(\Omega SU_{k+1})$ the induced map on homology. We have the commutative diagram

$$\begin{tikzcd} H_*(\Omega SU_k) \arrow[r, "j^{(k-1)}"] \arrow[d, "i_*^{(k)}"'] & \mathbb{Z}[h_1,h_2,\dots,h_{k-1}] \arrow[d, "\mathrm{incl}"] \\ H_*(\Omega SU_{k+1}) \arrow[r, "j^{(k)}"'] & \mathbb{Z}[h_1,h_2,\dots,h_k] \end{tikzcd} \tag{22}$$

Then

$$i_*^{(k)}([X_\mu]_*) = \sum_{\lambda \in \mathcal{C}^{k+1}} b_{\mu\lambda}^{(k)} [X_\lambda]_* \qquad \text{for } \mu \in \mathcal{C}^k. \tag{23}$$

It is shown using geometric techniques that $b_{\mu\lambda}^{(k)} \in \mathbb{Z}_{\geq 0}$ in [7].

This entire picture can be dualized. There is a Hopf algebra isomorphism [6]

$$\begin{aligned} H^*(\mathrm{Gr}_{\mathrm{SL}_{k+1}}) &\longrightarrow \Lambda/I_k \\ [X_\lambda]^* &\longmapsto \mathrm{Weak}^{(k)}_\lambda[X] \end{aligned} \tag{24}$$

mapping cohomology Schubert classes to dual k-Schur functions. Writing $i^{(k)*}$: $H^*(\Omega SU_{k+1}) \to H^*(\Omega SU_k)$ and $\pi^{(k)} : \Lambda/I_k \to \Lambda/I_{k-1}$ for the natural projection, we have the commutative diagram

$$\begin{tikzcd} H^*(\Omega SU_{k+1}) \arrow[r, "\cong"] \arrow[d, "i^{(k)*}"'] & \Lambda/I_k \arrow[d, "\pi^{(k)}"] \\ H^*(\Omega SU_k) \arrow[r, "\cong"'] & \Lambda/I_{k-1} \end{tikzcd} \tag{25}$$

$$i^{(k)*}([X_\lambda]^*) = \sum_{\mu\in\mathcal{C}^k} b^{(k)}_{\mu\lambda}[X_\mu]^* \qquad \text{for } \lambda \in \mathcal{C}^{k+1} \tag{26}$$

Using (24) and (26), one has

$$\pi^{(k)}(\mathrm{Weak}^{(k)}_\lambda[X]) = \sum_{\mu\in\mathcal{C}^k} b^{(k)}_{\mu\lambda}\,\mathrm{Weak}^{(k-1)}_\mu[X] \tag{27}$$

The combinatorics of this article is set in the cohomological side of the picture. However, we also speculate that the k-shape functions $\mathfrak{s}^{(k)}_\lambda[X]$ ($\lambda \in \Pi^k$) represent naturally-defined finite-dimensional subvarieties of $\mathrm{Gr}_{\mathrm{SL}_{k+1}}$, interpolating between the Schubert varieties of $\mathrm{Gr}_{\mathrm{SL}_{k+1}}$ and (the image in $\mathrm{Gr}_{\mathrm{SL}_{k+1}}$ of) the Schubert varieties of $\mathrm{Gr}_{\mathrm{SL}_k}$. Definition (18) would then express the decomposition of this subvariety in terms of Schubert classes in $H_*(\mathrm{Gr}_{\mathrm{SL}_{k+1}})$.

1.5. **k-branching polynomials and strong k-tableaux.** The results of this paper suggest an approach to proving Conjecture 3. Recall that the conjecture concerns the graded k-Schur functions $s^{(k)}_\lambda[X;t]$, for which there are several conjecturally equivalent characterizations. Our approach lends itself to proving the conjecture for the description of k-Schur functions given in [8]; that is, as the weight generating function of strong k-tableaux:

$$s^{(k)}_\lambda[X;t] = \sum_{T\in\mathrm{STab}^{k+1}_\lambda} x^{\mathrm{wt}(T)} t^{\mathrm{spin}(T)} \tag{28}$$

where $\mathrm{STab}^{k+1}_\lambda$ is the set of strong $(k+1)$-core tableaux of shape λ and $\mathrm{spin}(T)$ is a statistic assigned to strong tableaux. Note, it was shown [8] that the $s^{(k)}_\lambda$ used in this article equals the specialization of this function when $t = 1$.

To prove Conjecture 3, it suffices to give a bijection for each $\mu \in \mathcal{C}^k$:

$$\begin{aligned} \mathrm{STab}^k_\mu &\to \bigsqcup_{\lambda\in\mathcal{C}^{k+1}} \mathrm{STab}^{k+1}_\lambda \times \overline{\mathcal{P}}^k(\lambda,\mu) \\ U' &\mapsto (T', [\mathbf{p}]) \end{aligned} \tag{29}$$

such that

$$\mathrm{wt}(U') = \mathrm{wt}(T') \quad \text{and} \quad \mathrm{spin}(U') = \mathrm{spin}(T') + \mathrm{charge}(p). \tag{30}$$

To achieve this, the notion of strong strip (defined on cores) needs to be generalized to certain intervals $\mu \subset \lambda$ of k-shapes $\lambda, \mu \in \Pi^k$.

We should point out that the symmetry of the k-Schur functions defined by (28) is non-trivial. A forthcoming paper of Assaf and Billey [1] proves this result, as well as the positivity of $b_{\mu\lambda}^{k\to\infty}(t)$, using dual equivalence graphs. The bijection described above would also give a direct proof of the symmetry.

1.6. **Tableaux atoms and bijection** (20)**.** The earliest characterization of k-Schur functions is the tableaux atom definition of [9]. The definition has the form

$$s_\mu^{(k)}[X;t] = \sum_{T\in\mathbb{A}_\mu^{(k)}} t^{\mathrm{charge}(T)} s_{\mathrm{shape}(T)}\,, \tag{31}$$

where $\mathbb{A}_\mu^{(k)}$ is a certain set of tableaux constructed recursively using katabolism. It is immediate from the definition that

$$b_{\mu\lambda}^{k\to\infty}(t) = \sum_{\substack{T\in\mathbb{A}_\mu^{(k)}\\ \mathrm{shape}(T)=\lambda}} t^{\mathrm{charge}(T)}\,.$$

Unfortunately, actually determining which tableaux are in an atom $\mathbb{A}_\mu^{(k)}$ is an extremely intricate process.

Nonetheless, the construction of our bijection (20) was guided by the tableaux atoms and has led us to yet another conjecturally equivalent characterization for the k-Schur functions. In particular, iterating the bijection from a tableau T of weight $\mu\vdash n$, we get:

$$T\mapsto (T^{(n-1)},[\mathbf{p}_{n-1}]),\, T^{(n-1)}\mapsto (T^{(n-2)},[\mathbf{p}_{n-2}]),\ldots,\, T^{(k+1)}\mapsto (T^{(k)},[\mathbf{p}_k])\,. \tag{32}$$

Namely, this provides a bijection between T and $(T^{(k)},[\mathbf{p}_{n-1}],\ldots,[\mathbf{p}_k])$. We then say that $T^{(k)}$ is the k-tableau associated to T and conjecture that

Conjecture 6. *Let ρ be the unique element of $\mathcal{C}^{k+1}$ such that $\mathrm{rs}(\rho)=\mu$, and let $T_\mu^{(k)}$ be the unique k-tableau of weight μ and shape ρ (see* [11]*). Then*

$$\mathbb{A}_\mu^{(k)} = \Big\{ T \text{ of weight } \mu \,\Big|\, T_\mu^{(k)} \text{ is the } k\text{-tableau associated to } T \Big\}\,. \tag{33}$$

Support for this conjecture is given in [14] where it is shown that the bijection between T and $(T^{(k)},[\mathbf{p}_{n-1}],\ldots,[\mathbf{p}_k])$ is compatible with charge. In particular, it is shown that one can define a charge on k-tableaux satisfying the relation

$$\mathrm{charge}(T) = \mathrm{charge}(T^{(k)}) + \mathrm{charge}([\mathbf{p}_{n-1}]) + \cdots + \mathrm{charge}([\mathbf{p}_k])\,. \tag{34}$$

1.7. **Connection with representation theory.** In his thesis, L.-C. Chen [3] defined a family of graded S_n-modules associated to skew shapes whose row shape and column shape are partitions. Applying the Frobenius map (Schur-Weyl duality) to the characters of these modules, one obtains symmetric functions. Chen has a remarkable conjecture on their Schur expansions, formulated in terms of katabolizable tableaux. We expect that if the skew shape is the k-boundary of a k-shape λ then the resulting symmetric function is the homology k-shape function $\mathfrak{s}_\lambda[X;t]$ defined in (18). In [3], an important conjectural connection is also made between the above S_n-modules and certain virtual GL_n-modules supported in nilpotent conjugacy classes, via taking the zero weight space.

Using a subquotient of the extended affine Hecke algebra, J. Blasiak [2] constructed a noncommutative analogue of the Garsia-Procesi modules R_λ, whose Frobenius image is the modified Hall-Littlewood symmetric function. In this setup

there is an analogue of katabolizable tableaux and conjectured analogues of homology k-shape functions and the atoms of [9] and [3].

1.8. **Outline.** In §2 we define basic objects of interest here such as k-shapes, moves and the k-shape poset, and give some of their elementary properties. In §3 we introduce an equivalence relation on paths in the k-shape poset called diamond equivalence and show that it is generated by a smaller set of equivalences called elementary equivalences. In §4 we introduce covers and strips for k-shapes, and prove that there is a unique path in the k-shape poset allowing the extraction of a maximal strip from a given strip (Proposition 91). In §4 we also state the main result (bijection (20)) of this article (Theorem 75) and show how it leads to Theorem 4 and Theorem 5. Elementary properties of the functions $\mathfrak{S}_\lambda^{(k)}[X]$ and $\mathfrak{s}_\lambda^{(k)}[X]$ such as triangularity and conjugation are given in §§4.4.

The remaining sections, which contain the bulk of the technical details in this article, are concerned with the proof of bijection (20) by iteration of the *pushout*. This bijection sends compatible initial pairs (certain pairs (S, m) consisting of a strip S and a move m, both of which start from a common k-shape) to compatible final pairs (certain pairs (S', m') consisting of a strip S' and a move m', both of which end at a common k-shape). The basic properties of the pushout are established in §5 and §6. The most technical parts of this article (§7 and §8) are devoted to the interaction between pushouts and equivalences in the k-shape poset. The basic statement can be summarized as: *pushouts send equivalent paths to equivalent paths*. In §9-§14 we develop, in a brief form, the pullback, which is inverse to the pushout (§15).

For those interested in getting a quick hold on the pushout algorithm on which bijection (20) relies, we suggest reading the beginning of §2, §3, §4 and §7 to get the basic definitions and ideas, along with §§4.9 and §§7.1 that describe canonical processes to obtain a maximal strip and to perform the pushout respectively.

Contents

Acknowledgments. This project was supported by NSF grants DMS-0638625, DMS-0652641, DMS-0652648, DMS-0652668, and DMS-0901111. T.L. was supported by a Sloan Fellowship. L.L was supported by FONDECYT (Chile) grant #1090016 and by CONICYT (Chile) grant ACT56 Lattices and Symmetry.

This project benefited from discussions with Jason Bandlow, Francois Descouens, Florent Hivert, Anne Schilling, Nicolas Thiéry, and Mike Zabrocki.

2. The k-shape poset

For a fixed positive integer k, the object central to our study is a family of "k-shape" partitions that contains both k and $k+1$-cores. The formula for k-branching coefficients counts paths in a poset on the k-shapes. As with Young order, we will define the order relation in terms of adding boxes to a given vertex λ, but now the added boxes must form a sequence of "strings". Here we introduce k-shapes, strings, and moves – the ingredients for our poset.

2.1. **Partitions.** Let $\mathbb{Y} = \{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \in \mathbb{Z}_{\geq 0}^\infty \mid \lambda_i = 0 \text{ for } i \gg 0\}$ denote the set of partitions. Each $\lambda \in \mathbb{Y}$ can be identified with its Ferrers diagram $\{(i,j) \in \mathbb{Z}_{>0}^2 \mid j \leq \lambda_i\}$. The elements of $\mathbb{Z}_{>0}^2$ are called cells. The row and column indices of a cell $b = (i,j)$ are denoted $\mathrm{row}(b) = i$ and $\mathrm{col}(b) = j$. We use the French/transpose-Cartesian depiction of $\mathbb{Z}_{>0}^2$: row indices increase from bottom to top. The transpose involution on $\mathbb{Z}_{>0}^2$ defined by $(i,j) \mapsto (j,i)$ induces an involution on $\mathbb{Y}$ denoted $\lambda \mapsto \lambda^t$. The *diagonal index* of $b = (i,j)$ is given by $d(b) = j - i$ and we then define the *distance* between cells x and y to be $|d(x) - d(y)|$.

The arm (resp. leg) of $b = (i,j) \in \lambda$ is defined by $a_\lambda(b) = \lambda_i - j$ (resp. $l_\lambda(b) = \lambda_j^t - i$) is the number of cells in the diagram of λ in the row of b to its right (resp. in the column of b and above it). The hook length of $b = (i,j) \in \lambda$ is defined by $h_\lambda(b) = a_\lambda(b) + l_\lambda(b) + 1$. Let $\mathcal{C}^k = \{\lambda \in \mathbb{Y} \mid h_\lambda(b) \neq k \text{ for all } b \in \lambda\}$ be the set of k-cores.

Let $D = \mu/\lambda$ be a skew shape, the difference of Ferrers diagrams of partitions $\mu \supset \lambda$. Although such a set of cells may be realized by different pairs of partitions, unless specifically stated otherwise, we shall use the notation μ/λ with the fixed pair $\lambda \subset \mu$ in mind. D is referred to as λ-*addable* and μ-*removable*. A *horizontal* (resp. *vertical*) strip is a skew shape that contains at most one cell in each column (resp. row). A λ-addable cell (corner) is a skew shape μ/λ consisting of a single cell. Define $\mathrm{top}_c(D)$ and $\mathrm{bot}_c(D)$ to be the top and bottom cells in column c of D and let $\mathrm{right}_r(D)$ and $\mathrm{left}_r(D)$ be the rightmost and leftmost cells in row r of D. Let c^+ (resp. c^-) denote the column right-adjacent (resp. left-adjacent) to column c. Similar notation is used for rows.

2.2. **k-shapes.** The k-interior of a partition λ is the subpartition of cells with hook length exceeding k:

$$\mathrm{Int}^k(\lambda) = \{b \in \lambda \mid h_\lambda(b) > k\}.$$

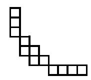
The *k-boundary* of λ is the skew shape of cells with hook bounded by k:

$$\partial^k(\lambda) = \lambda/\mathrm{Int}^k(\lambda).$$

We define the *k-row shape* $\mathrm{rs}^k(\lambda) \in \mathbb{Z}_{\geq 0}^{\infty}$ (resp. *k-column shape* $\mathrm{cs}^k(\lambda) \in \mathbb{Z}_{\geq 0}^{\infty}$) of λ to be the sequence giving the numbers of cells in the rows (resp. columns) of $\partial^k(\lambda)$.

Definition 7. A partition λ is a *k-shape* if $\mathrm{rs}^k(\lambda)$ and $\mathrm{cs}^k(\lambda)$ are partitions. Π^k denotes the set of k-shapes and $\Pi^k_N = \{\lambda \in \Pi^k : |\partial^k(\lambda)| = N\}$.

Example 8. $\lambda = (8,4,3,2,1,1,1) \in \Pi^4_{12}$, since $\mathrm{rs}^4(\lambda) = (4,2,2,1,1,1,1)$ and $\mathrm{cs}^4(\lambda) = (3,2,2,1,1,1,1,1)$ are partitions and $|\partial^4(\lambda)| = 4+2+2+1+1+1+1 = 12$. $\mu = (3,3,1) \notin \Pi^4$ since $\mathrm{rs}^4(\mu) = (2,3,1)$ is not a partition.



$\partial^4(\lambda)$

$\partial^4(\mu)$

Remark 9. The transpose map is an involution on Π^k_N.

The set of k-shapes includes both the k-cores and $k+1$-cores.

Proposition 10. *$\mathcal{C}^k \subset \Pi^k$ and $\mathcal{C}^{k+1} \subset \Pi^k$.*

Proof. It is shown in [11] that
$$\lambda \mapsto \mathrm{rs}^k(\lambda) \tag{35}$$
is a bijection from $\mathcal{C}^{k+1} \to \mathcal{B}^k$ implying that $\mathrm{rs}^k(\lambda) \in \mathbb{Y}$. Similarly, $\lambda \mapsto \mathrm{rs}^k(\lambda^t) = \mathrm{cs}^k(\lambda)$ is a bijection, and thus $\mathcal{C}^{k+1} \subset \Pi^k$. In particular $\mathcal{C}^k \subset \Pi^{k-1}$. For $\lambda \in \mathcal{C}^k$ we have $\partial^k(\lambda) = \partial^{k-1}(\lambda)$, from which it follows that $\lambda \in \Pi^k$. $\square$

Since $k \geq 2$ remains fixed throughout, we shall often suppress k in the notation, writing $\partial\lambda$, $\mathrm{rs}(\lambda)$, $\mathrm{cs}(\lambda)$, Π, and so forth.

Remark 11. A k-shape λ is uniquely determined by its row shape $\mathrm{rs}(\lambda)$ and column shape $\mathrm{cs}(\lambda)$.

Remark 12. Consider a partition λ with addable corners x and y in columns c and c^+, respectively. If $h_\lambda(\mathrm{left}_{\mathrm{row}(x)}(\partial\lambda)) = k$ then $\mathrm{rs}(\lambda)_{\mathrm{row}(x)} = \mathrm{rs}(\lambda)_{\mathrm{row}(y)}$ since the cell below $\mathrm{left}_{\mathrm{row}(x)}(\partial\lambda)$ is not in $\partial\lambda$.

Remark 13. Suppose for some $c, p \geq 1$ and $\mu \in \Pi$, the cells $\mathrm{top}_j(\partial\mu)$ for $c \leq j < c+p$, all lie in the same row. As $\mathrm{cs}(\mu)$ and $\mathrm{Int}(\mu)$ are partitions, it follows that the cells $\mathrm{bot}_j(\partial\mu)$ lie in the same row (say the r-th) for $c \leq j < c+p$. Since $\mathrm{rs}(\mu)$ is a partition, one may deduce that $\mu_{r-1} \geq \mu_r + p$. In particular, there is a μ-addable corner in the row of $\mathrm{bot}_c(\partial\mu)$ for all columns c.

2.3. **Strings.** Given the k-shape vertices, the primary notion to define our order is a string of cells lying at a diagonal distance k or $k+1$ from one another. To be precise, let b and b' be *contiguous* cells when $|d(b) - d(b')| \in \{k, k+1\}$.

Remark 14. Since λ-addable cells occur on consecutive diagonals, a λ-addable corner x is contiguous with at most one λ-addable corner above (resp. below) it.

Definition 15. A *string* of *length* ℓ is a skew shape μ/λ which consists of cells $\{a_1, \ldots, a_\ell\}$, where a_{i+1} is below a_i and they are contiguous for each $1 \leq i < \ell$.

Note that all cells in a string $s = \mu/\lambda$ are λ-addable and μ-removable. We thus refer to λ-addable or μ-removable strings. Any string $s = \mu/\lambda$ can be categorized into one of four types depending on the elements of $\partial\lambda \setminus \partial\mu$, as described by the following property.

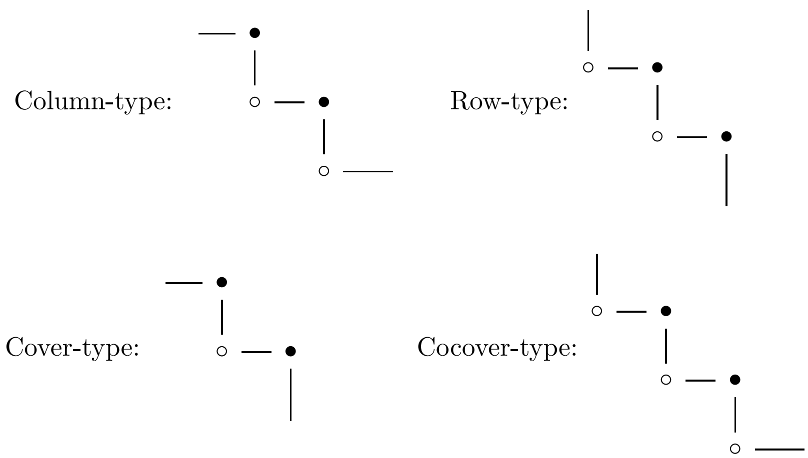


FIGURE 1. Types of string diagrams

Property 16. *For any string $s = \mu/\lambda = \{a_1, a_2, \ldots, a_\ell\}$, let $b_0 = \mathrm{left}_{\mathrm{row}(a_1)}(\partial\lambda)$, $b_\ell = \mathrm{bot}_{\mathrm{col}(a_\ell)}(\partial\lambda)$, and $b_i = (\mathrm{row}(a_{i+1}), \mathrm{col}(a_i))$ for $1 \leq i < \ell$.*

$$\partial\lambda \setminus \partial\mu = \begin{cases} \{b_1, \ldots, b_{\ell-1}\} & \text{if } h_\lambda(b_0) < k \text{ and } h_\lambda(b_\ell) < k \\ \{b_0, b_1, \ldots, b_{\ell-1}\} & \text{if } h_\lambda(b_0) = k \text{ and } h_\lambda(b_\ell) < k \\ \{b_1, \ldots, b_{\ell-1}, b_\ell\} & \text{if } h_\lambda(b_0) < k \text{ and } h_\lambda(b_\ell) = k \\ \{b_0, b_1, \ldots, b_{\ell-1}, b_\ell\} & \text{if } h_\lambda(b_0) = h_\lambda(b_\ell) = k \end{cases} \tag{36}$$

Proof. For any $1 \leq i < \ell$, $b_i \in \partial\lambda \setminus \partial\mu$ by definition of contiguous. Otherwise, $\mathrm{left}_{\mathrm{row}(a_1)}(\partial\lambda)$ and $\mathrm{bot}_{\mathrm{col}(a_\ell)}(\partial\lambda)$ are the only other cells whose hooks may be k-bounded in λ and exceed k in μ. $\square$

Definition 17. A string $s = \mu/\lambda$ is defined to be one of four types, *cover-type, row-type, column-type*, or *cocover-type* when $\partial\lambda \setminus \partial\mu$ equals the first, second, third, or fourth set, respectively, given in (36).

It is helpful to depict a string $s = \mu/\lambda$ by its *diagram*, defined by the following data: cells of s are represented by the symbol $\bullet$, cells of $\partial\lambda \setminus \partial\mu$ a represented by $\circ$, and cells of $\partial\mu \cap \partial\lambda$ in the same row (resp. column) as some $\bullet$ or $\circ$ are collectively depicted by a horizontal (resp. vertical) line segment. The four possible string diagrams are shown in Figure 1.

Given a string $s = \mu/\lambda = \{a_1, \ldots, a_\ell\}$, of particular importance are the columns and rows in its diagram that contain only a $\circ$ or only $\bullet$. To precisely specify such rows and columns, we need some notation. For a skew shape $D = \mu/\lambda$, define $\Delta_{\mathrm{rs}}(D) = \mathrm{rs}(\mu) - \mathrm{rs}(\lambda) \in \mathbb{Z}^\infty$. The *positively* (resp. *negatively*) *modified rows* of D are those corresponding to positive (resp. negative) entries in $\Delta_{\mathrm{rs}}(D)$. Similar definitions apply for columns. It is clear from the Figure 1 diagrams that a given string has at most one positively or negatively modified row and column. Such rows and columns are earmarked as follows, given they exist:

- $c_{s,u}$ is the unique column negatively modified by s. Equivalently, $c_{s,u} = \mathrm{col}(\mathrm{left}_{\mathrm{row}(a_1)}(\partial\lambda))$ if and only if the leftmost column in the diagram of s has a $\circ$
- $r_{s,d}$ is the unique row negatively modified by s. Equivalently, $r_{s,d} = \mathrm{row}(\mathrm{bot}_{\mathrm{col}(a_\ell)})$ iff the lowest row in the diagram of s has a $\circ$
- $r_{s,u}$ is the unique row positively modified by s. Equivalently, $r_{s,u} = \mathrm{row}(a_1)$ iff the topmost row in the diagram of s has no $\circ$
- $c_{s,d}$ is the unique column positively modified by s. Equivalently, $c_{s,d} = \mathrm{col}(a_\ell)$ if the rightmost column in the diagram of s has no $\circ$.

Note that $c_{s,u} < \mathrm{col}(a_1)$ and $r_{s,d} < \mathrm{row}(a_\ell)$ when defined.

Remark 18. For a λ-addable string s, we have the following vector equalities in the free $\mathbb{Z}$-module $\mathbb{Z}^\infty = \bigoplus_{i\in\mathbb{Z}_{>0}} \mathbb{Z}e_i$ with standard basis $\{e_i \mid i \in \mathbb{Z}_{>0}\}$:

$$\Delta_{\mathrm{cs}}(s) = e_{c_{s,d}} - e_{c_{s,u}} \tag{37}$$

$$\Delta_{\mathrm{rs}}(s) = e_{r_{s,u}} - e_{r_{s,d}} \tag{38}$$

where by convention $e_i = 0$ if the subscript i is not defined (e.g. $c_{s,u}$ is not defined when $\mathrm{left}_{\mathrm{row}(a_1)}(\partial\lambda) \notin \partial\lambda \setminus \partial\mu$).

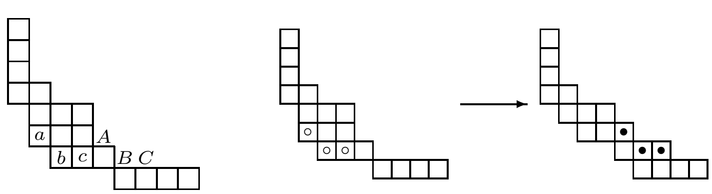
2.4. **Moves.** Our poset will be defined by taking a k-shape μ to be larger than $\lambda \in \Pi$ when the skew diagram μ/λ is a particular succession of strings (called a move). To this end, define two strings to be *translates* when they are translates of each other in $\mathbb{Z}^2$ by a fixed vector, and their corresponding modified rows and columns agree in size. Equivalently, their diagrams have the property that $\bullet$'s and $\circ$'s appear in the same relative positions with respect to each other and the lengths of each corresponding horizontal and vertical segment are the same. We will also refer to cells a_j and b_j as translates when strings $s_1 = \{a_1, \ldots, a_\ell\}$ and $s_2 = \{b_1, \ldots, b_\ell\}$ are translates.

Definition 19. A *row move* m of rank r and length ℓ is a chain of partitions $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \mu$ that meets the following conditions:

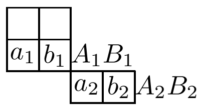
(1) $\lambda \in \Pi$
(2) $s_i = \lambda^i/\lambda^{i-1}$ is a row-type string consisting of ℓ cells for all $1 \le i \le r$
(3) the strings s_i are translates of each other
(4) the top cells of $s_1, \ldots, s_r$ occur in consecutive columns from left to right
(5) $\mu \in \Pi$.

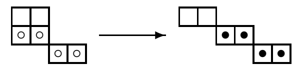
We say that m is a row move from λ to μ and write $\mu = m * \lambda$ or $m = \mu/\lambda$. A *column move* is the transpose analogue of a row move. A *move* is a row move or column move.

Example 20. For $k = 5$, a row move of length 1 and rank 3 with strings $s_1 = \{A\}$, $s_2 = \{B\}$, and $s_3 = \{C\}$ is pictured below. The lower case letters are the cells that are removed from the k-boundary when the corresponding strings are added.



For $k = 3$, a row move of length 2 and rank 2 with strings $s_1 = \{A_1, A_2\}$ and $s_2 = \{B_1, B_2\}$ is:





Note that a row move from λ to μ merits its name because $\partial\mu$ can be viewed as a right-shift of some rows of $\partial\lambda$. In particular, $|\partial\mu| = |\partial\lambda|$.

Property 21. *If a row move negatively (resp. positively) modifies a column then it negatively (resp. positively) modifies all columns of the same size to the right (resp. left).*

Proof. All of the columns positively (resp. negatively) modified by a row move, are consecutive and have the same size, by Definition 19(3),(4). The result follows from Definition 19(5). □

A move m is said to be *degenerate* if $c^+_{s_r,u} = c_{s_1,d}$. Note that a degenerate move can be of any rank but always has length 1. The first move in Example 20 is degenerate.

Property 22. *Condition* (5) *of Definition 19 is equivalent to*

- $\mathrm{cs}(\lambda)_{c_{s_r,u}} > \mathrm{cs}(\lambda)_{c^+_{s_r,u}}$ *and* $\mathrm{cs}(\lambda)_{c^-_{s_1,d}} > \mathrm{cs}(\lambda)_{c_{s_1,d}}$ *if m is nondegenerate.*
- $\mathrm{cs}(\lambda)_{c_{s_r,u}} > \mathrm{cs}(\lambda)_{c_{s_1,d}} + 1$ *if m is degenerate.*

Proof. The precise column and row modification of a string is pinpointed in Remark 18 and immediately implies the claim by definition of k-shape. □

Remark 23. Consider a k-shape λ and a string $s_1 = \lambda^1/\lambda$. If there is a row move from λ starting as $\lambda \subset \lambda^1$, then Conditions (3),(4) and (5) of Definition 19 determine $s_2, \ldots, s_r$ (and thus the move). Note that Property 21 and Property 22 ensures a unique r since $\mathrm{cs}(\lambda)_{c_{s_r,u}} > \mathrm{cs}(\lambda)_{c^+_{s_r,u}}$ implies that an extra row type string would not be a translate of s_1.

Lemma 24. *Suppose s and t are strings in a move m and the cells $x \in s$ and $y \in t$ are translates of each other. Then $|d(x) - d(y)| < k - 1$.*

Proof. Let $m = s_1 \cup \cdots \cup s_r$ be a row move from λ to μ and let $s_j = \{a_1^j, \ldots, a_n^j\}$ for $j = 1, \ldots, r$. It suffices to prove the case where $x = a_1^1$ and $y = a_1^r$ are the topmost cells of s_1 and s_r, respectively. First suppose that $d(a_1^r) - d(a_1^1) \geq k$. Then $c_{s_r,u} \geq \mathrm{col}(a_1^1)$ since $a_1^1, a_1^r \in \mu$, and further, $c_{s_r,u} < \mathrm{col}(a_1^r)$. Thus $c_{s_r,u} = \mathrm{col}(a_1^j)$ for some $j < r$ since $a_1^1, \ldots, a_1^r$ occur in adjacent columns by Definition 19 of row move. Moreover, m is a row move implies that s_j and s_r are translates and therefore $\mathrm{col}(a_1^j) = c_{s_r,u}$ of $\partial(\lambda \cup s_1 \cup \cdots \cup s_j)$ has the same length as $\mathrm{col}(a_1^r)$ in $\partial(\lambda \cup s_1 \cup \cdots \cup s_r = \mu)$. However, column $c_{s_r,u}$ is negatively modified by s_r implying the contradiction $\mu \notin \Pi$. In the case that $d(a_r^1) - d(a_1^1) = k - 1$, the top cell in column $c_{s_r,u}$ of $\partial(\lambda \cup s_1 \cup \cdots \cup s_{r-1})$ is left-adjacent to a_1^1. However, this column is negatively modified by s_r implying that in $\partial\mu$, it is shorter than $\mathrm{col}(a_1^1)$. Again, the assumption that $\mu \in \Pi$ is contradicted. □

Corollary 25. *The rank of a move is at most $k - 1$.*

Property 26.

(1) *If m is a row move where $\mu = m * \lambda$, then μ/λ is a horizontal strip*
(2) *If M is a column move where $\mu = M * \lambda$, then μ/λ is a vertical strip*
(3) *Any cell common to a row and a column move from the same shape λ, is a λ-addable corner.*

Proof. Consider a row move m from λ to μ with strings $s_1, s_2, \dots, s_r$ and let $s_1 = \{a_1, a_2, \dots, a_\ell\}$. Suppose that μ/λ is not a horizontal strip. Since the strings are translates and their topmost cells occur in consecutive columns by the definition of move, a violation of the horizontal strip condition must occur where a_2 lies below the top cell b_1 of string s_i, for some $i > 1$. Therefore, $|d(a_1) - d(b_1)| \in \{k-1, k\}$ since the definition of string implies $|d(a_1)-d(a_2)| \in \{k, k+1\}$. However, Lemma 24 is contradicted implying μ/λ is a horizontal strip. By the transpose argument, we also have that a column move is a vertical strip. (1) and (2) imply (3). □

Proposition 27. *Let m be a row or column move from λ to μ. Then the decomposition of $m = \mu/\lambda$ into strings (according to Definition 19) is unique.*

Proof. Given row move m from λ to μ, Remark 23 implies it suffices to show that the λ-addable string s_1 is uniquely determined. By (37) and Definition 19(3), for any $m = \mu/\lambda = \{s_1, \dots, s_r\}$,

$$\Delta_{\mathrm{cs}}(\mu/\lambda) = \sum_{j=0}^{r-1} (e_{c_{s_1,d}+j} - e_{c_{s_1,u}+j}). \tag{39}$$

Since $c_{s_r,u} < \mathrm{col}(a_1) \leq \mathrm{col}(a_\ell) = c_{s_1,d}$ there is no cancellation in this formula, so the rank of m can be read from the number of consecutive $+1$'s in $\Delta_{\mathrm{cs}}(\mu/\lambda)$ (and is independent of $s_1, \dots, s_r$). The length of m is then simply $|\mu/\lambda|/r$. Since the leftmost cell of the horizontal strip m must be the top cell of the first string s of m and the length of s is determined, by Remark 14 it follows that the λ-addable string s_1 is determined. □

2.5. **Poset structure on k-shapes.** We endow the set Π_N of k-shapes of fixed size N, with the structure of a directed acyclic graph with an edge from λ to μ if there is a move from λ to μ. Since a row (resp. column) move from λ to μ satisfies $\mathrm{rs}(\lambda) = \mathrm{rs}(\mu)$ and $\mathrm{cs}(\lambda) \trianglerighteq \mathrm{cs}(\mu)$ (resp. $\mathrm{cs}(\lambda) = \mathrm{cs}(\mu)$ and $\mathrm{rs}(\lambda) \trianglerighteq \mathrm{rs}(\mu)$), this directed graph induces a poset structure on Π_N which is a subposet of the Cartesian square of the dominance order $\trianglerighteq$ on partitions of size N.

Proposition 28. *An element of the k-shape poset is maximal (resp. minimal) if and only if it is a $(k+1)$-core (resp. k-core).*

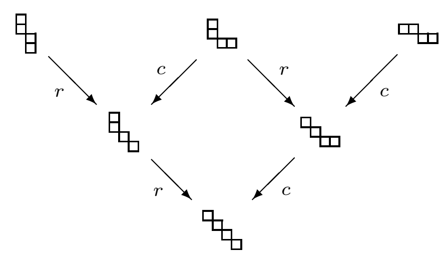
Proof. Since a k-core has no hook sizes of size k, it also has no row-type or column-type strings addable. Thus k-cores are minimal elements of the k-shape poset. Now suppose λ is a minimal element of the k-shape poset, and suppose λ has a hook of size k. Let us take the rightmost such cell of $\partial\lambda$, say b. Then there is a λ-addable corner a_1 at the end of the row of b. Let $s = \{a_1, a_2, \dots, a_\ell\}$ be the longest row-type string with top cell a_1 (see Lemma 30).

Suppose $\mathrm{cs}(\lambda)_{\mathrm{col}(b)} = \mathrm{cs}(\lambda)_{\mathrm{col}(b)+1} = \cdots = \mathrm{cs}(\lambda)_{\mathrm{col}(b)+t} > \mathrm{cs}(\lambda)_{\mathrm{col}(b)+t+1}$. Then the bottom cells in $\partial\lambda$ of columns $\mathrm{col}(b), \mathrm{col}(b)+1, \dots, \mathrm{col}(b)+t+1$ all lie in the same row as b, for otherwise such a cell would have a hook-length of size k (or λ would not be a k-shape). Since λ is a k-shape, there are t successively addable cells to the right

of a_1, on the same row as a_1. A similar argument shows that we can in fact find $t+1$ row-type strings $s = s_1, s_2, \ldots, s_{t+1}$ whose cells are on exactly the same set of rows and which have identical diagrams. We claim that $m = s_1 \cup s_2 \cup \cdots \cup s_{t+1}$ is a row move on λ. Let $\mu = m * \lambda$. By construction, $\mathrm{cs}(\mu)_{\mathrm{col}(b)+t} \geq \mathrm{cs}(\mu)_{\mathrm{col}(b)+t+1}$. It thus suffices to show that $\mathrm{cs}(\mu)_{\mathrm{col}(a_\ell)-1} \geq \mathrm{cs}(\mu)_{\mathrm{col}(a_\ell)}$. Since s is row-type, and cannot be extended further below, the cell $d = \mathrm{bot}_{\mathrm{col}(a_\ell)}(\partial\lambda)$ has hook length $< k-1$. Suppose $\mathrm{cs}(\lambda)_{\mathrm{col}(a_\ell)-1} = \mathrm{cs}(\lambda)_{\mathrm{col}(a_\ell)}$. Since a_ℓ is λ-addable, the bottom cell c of column $\mathrm{col}(a_\ell) - 1$ in $\partial\lambda$ must be above the bottom of column $\mathrm{col}(a_\ell)$. But the cell c' directly below c has hook length $h_\lambda(c') \leq h_\lambda(d) + 2 < k+1$. This is a contradiction.

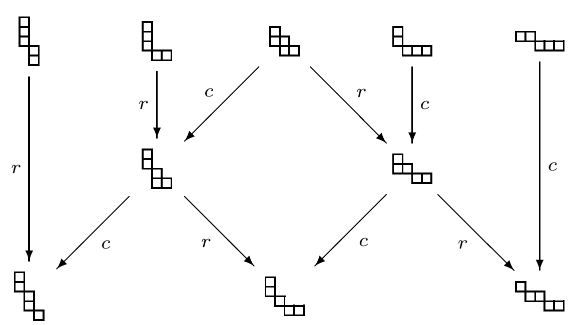
The proof that the $(k+1)$-cores are exactly the maximal elements is similar. □

Example 29. The graph Π_4^2 is pictured below. Only the cells of the k-boundaries are shown. Row moves are indicated by r and column moves by c.



(40)

The graph Π_5^3 is pictured below.



(41)

2.6. **String and move miscellany.** Here we highlight a number of lemmata about strings that will be needed later.

In the special case that μ or λ is a k-shape, the string $s = \mu/\lambda$ obeys a number of explicit properties.

Lemma 30. *Let $\lambda \in \Pi$ and $s = \{a_1, \ldots, a_\ell\}$ be a λ-addable string.*

(1) *If s negatively modifies a row, then it can be extended below to a λ-addable string that does not have negatively modified rows.*
(2) *If s negatively modifies a column, then it can be extended above to a λ-addable string that does not have negatively modified columns.*

Proof. Let s negatively modify a row. By Remark 13, there is a λ-addable cell x in the row of $b = \mathrm{bot}_{\mathrm{col}(a_\ell)}(\partial\lambda)$ and we have $h_\lambda(b) = k$. Therefore $d(x) - d(a_\ell) = k+1$ and $s \cup \{x\}$ is a λ-addable string that extends s below. The required string exists by induction. Part (2) is similar. □

Lemma 31. *Let $\lambda \in \Pi$ and $s = \{a_1, \ldots, a_\ell\}$ be a λ-addable string.*

(1) $\lambda_{\mathrm{row}(a_j)-1} - \lambda_{\mathrm{row}(a_j)} \geq \lambda_{\mathrm{row}(a_i)-1} - \lambda_{\mathrm{row}(a_i)}$ *for all* $i < j$,
(2) $\lambda^t_{\mathrm{col}(a_j)-1} - \lambda^t_{\mathrm{col}(a_j)} \geq \lambda^t_{\mathrm{col}(a_i)-1} - \lambda^t_{\mathrm{col}(a_i)}$ *for all* $j > i$,

with the convention that $\lambda_{\mathrm{row}(a_\ell)-1}$ (resp. $\lambda^t_{\mathrm{col}(a_1)-1}$) is infinite if a_ℓ (resp. a_1) lies in the first row (resp. column) of λ.

Proof. Part (1) follows from Remark 13 and Part (2) follows by transposition. □

Lemma 32. *Let $s = \{a_1, \ldots, a_n\}$ and $t = \{b_i, b_{i+1}\}$ be λ-addable strings for some $\lambda \in \Pi$. If $\mathrm{row}(b_i) = \mathrm{row}(a_j) + 1$ and $\mathrm{row}(b_{i+1}) = \mathrm{row}(a_{j+1}) + 1$ (or $\mathrm{col}(b_i) = \mathrm{col}(a_j)+1$ and $\mathrm{col}(b_{i+1}) = \mathrm{col}(a_{j+1})+1$), then $|d(b_i) - d(b_{i+1})| \leq |d(a_j) - d(a_{j+1})|$. This also holds if s and t are λ-removable.*

Proof. Note that for any x contiguous to and higher than x', $\mathrm{cs}(\lambda)_{\mathrm{col}(x)} = \mathrm{row}(x) - \mathrm{row}(x')$ and $|d(x) - d(x')| = \mathrm{cs}(\lambda)_{\mathrm{col}(x)} + \mathrm{rs}(\lambda)_{\mathrm{row}(x')}$. Thus, $\mathrm{cs}(\lambda)_{\mathrm{col}(b_i)} = \mathrm{cs}(\lambda)_{\mathrm{col}(a_j)}$. Since $\lambda \in \Pi$ implies that $\mathrm{rs}(\lambda)_{\mathrm{row}(b_{i+1})} \leq \mathrm{rs}(\lambda)_{\mathrm{row}(a_{j+1})}$ we then have our claim. □

Lemma 33. *Let $\lambda \in \Pi$. Consider a λ-addable corner b and some $x \notin \lambda$ in a lower row than* $\mathrm{row}(b)$ *that is right-adjacent to a cell in λ. If $|d(b) - d(x)| = k + 1$ then x is λ-addable and if $|d(b) - d(x)| = k$ then either x or the cell immediately below x is λ-addable.*

Proof. When $|d(b) - d(x)| = k+1$, $\mathrm{bot}_{\mathrm{col}(b)}(\partial\lambda)$ is in the row of x and thus Remark 13 implies that $\mathrm{row}(x)$ has λ-addable corner (namely x). If $|d(b) - d(x)| = k$, then either $\mathrm{bot}_{\mathrm{col}(b)}(\partial\lambda)$ is in the row of x (and as before, x is λ-addable) or $\mathrm{bot}_{\mathrm{col}(b)}(\partial\lambda)$ is in the row below x. If x is not λ-addable then the latter case holds and the cell immediately below x is λ-addable. □

Lemma 34. *Let m be a row or column move from λ to μ. For any cells $a, b \in m$ that are translates of each other,* $\mathrm{rs}(\lambda)_{\mathrm{row}(a)} = \mathrm{rs}(\lambda)_{\mathrm{row}(b)}$, $\mathrm{cs}(\lambda)_{\mathrm{col}(a)} = \mathrm{cs}(\lambda)_{\mathrm{col}(b)}$, $\mathrm{rs}(\mu)_{\mathrm{row}(a)} = \mathrm{rs}(\mu)_{\mathrm{row}(b)}$ *and* $\mathrm{cs}(\mu)_{\mathrm{col}(a)} = \mathrm{cs}(\mu)_{\mathrm{col}(b)}$.

Proof. Consider the case that m is a row move (the column case follows by transposition). By definition of move, the strings of m have diagrams which are translates of each other. Since Property 26 implies the strings never lie on top of each other, if cell a is the translate of cell b then $\mathrm{cs}(\lambda)_{\mathrm{col}(a)} = \mathrm{cs}(\lambda)_{\mathrm{col}(b)}$ and $\mathrm{cs}(\mu)_{\mathrm{col}(a)} = \mathrm{cs}(\mu)_{\mathrm{col}(b)}$. Since strings in a row move never change the row reading we have by translation of diagrams that $\mathrm{rs}(\lambda)_{\mathrm{col}(a)} = \mathrm{rs}(\lambda)_{\mathrm{col}(b)}$ and $\mathrm{rs}(\mu)_{\mathrm{col}(a)} = \mathrm{rs}(\mu)_{\mathrm{col}(b)}$. □

Let b be a cell in a skew shape D. Define the *indent* of b in D by $\mathrm{Ind}_D(b) = \mathrm{col}(b) - \mathrm{col}(\mathrm{left}_{\mathrm{row}(b)}(D))$; this is the number of cells strictly to the left of b in its row in D. If D is a horizontal strip and $b \in D$ then b is λ-addable if and only if $\mathrm{Ind}_D(b) = 0$.

Lemma 35. *Let $\lambda \in \Pi$, m a row move from λ, and s a string of m. Then $\mathrm{Ind}_m(b)$ is constant for $b \in s$. In particular, if some cell of m is λ-addable, then so is every cell in its string.*

Proof. The first assertion follows by induction on $\mathrm{Ind}_m(b)$ using Definition 19(3). The second holds by Proposition 27. □

3. Equivalence of paths in the k-shape poset

3.1. **Diamond equivalences.**

Definition 36. Given a move m, the *charge* of m, written $\mathrm{charge}(m)$, is 0 if m is a row move and $r\ell$ if m is a column move of length ℓ and rank r. Notice that in the column case, $r\ell$ is simply the number of cells in the move m when viewed as a skew shape. The charge of a path $(m_1, \ldots, m_n)$ in Π_N is $\mathrm{charge}(m_1) + \cdots + \mathrm{charge}(m_n)$, the sum of the charges of the moves that constitute the path.

Let $\equiv$ be the equivalence relation on directed paths in Π_N generated by the following *diamond equivalences*:

$$\tilde{M} m \equiv \tilde{m} M \tag{42}$$

where $m, M, \tilde{m}, \tilde{M}$ are moves (possibly empty) between k-shapes such that the diagram

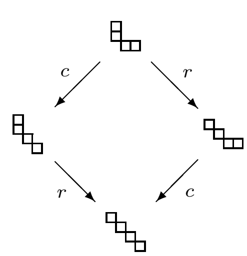
$$\begin{tikzcd} & \lambda \arrow[dl, "m"'] \arrow[dr, "M"] & \\ \mu \arrow[dr, "\tilde{M}"'] & & \nu \arrow[dl, "\tilde{m}"] \\ & \gamma & \end{tikzcd} \tag{43}$$

commutes and the charge is the same on both sides of the diamond:

$$\mathrm{charge}(m) + \mathrm{charge}(\tilde{M}) = \mathrm{charge}(M) + \mathrm{charge}(\tilde{m}). \tag{44}$$

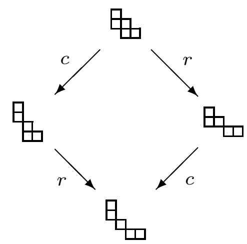
The commutation is equivalent to the equality $\tilde{M} \cup m = \tilde{m} \cup M$ where a move is regarded as a set of cells. Observe that the charge is by definition constant on equivalence classes of paths.

Example 37. Continuing Example 29, the two paths in Π_4^2 from $\lambda = (3,1,1)$ to $\nu = (4,3,2,1)$ have charge 2 and 3 respectively, and so are not equivalent. Thus by Theorem 2 one has $b_{\mu\lambda}^{(k)} = 2$, and according to Conjecture 3, we have $b_{\mu\lambda}^{(k)} = t^2 + t^3$.



(45)

The two paths in Π_5^3 from $\lambda = (3,2,1)$ to $\nu = (4,2,1,1)$ are diamond equivalent, both having charge 1. Thus by Theorem 2 one has $b_{\mu\lambda}^{(k)} = 1$, and according to Conjecture 3, we have $b_{\mu\lambda}^{(k)} = t$.



(46)

We will describe in more detail in this section when two moves m and M can obey a diamond equivalence. We will also see that the relation $\equiv$ is generated by special diamond equivalences called *elementary equivalences* (see Proposition 55).

3.2. **Elementary equivalences.** We require a few more notions to define elementary equivalence.

Let m and M be moves from $\lambda \in \Pi$. We say that m and M *intersect* if they are non-disjoint as sets of cells. Similarly, we say that two strings s and t intersect if they have cells in common. We say that the pair (m, M) is *reasonable* if for every string s and t of m and M respectively that intersect, we either have $s \subseteq t$ or $t \subseteq s$.

Let s and t be intersecting strings. Either the highest (resp. lowest) cell of $s \cup t$ is in $s \setminus t$, or in $s \cap t$, or in $t \setminus s$; in these cases we say that s continues above (resp. below) t, or s and t are matched above (resp. below), or t continues above (resp. below) s. We say that m continues above (resp. below) M, or m and M are matched above (resp. below), or M continues above (resp. below) m, if the corresponding relation holds for all pairs of strings s in m and t in M such that $s \cap t \neq \emptyset$.

We say that the disjoint strings s and t are *contiguous* if $s \cup t$ is a string. We say that the moves m and M are not contiguous if no string of m is contiguous to a string of M.

For the sake of clarity, the overall picture is presented first, the proofs being relegated to Subsections 3.8, 3.9 and 3.10.

The following lemma asserts that any pair of intersecting strings $s \subset m$ and $t \subset M$ are in the same relative position.

Lemma 38. *Let m and M be intersecting λ-addable moves for $\lambda \in \Pi$. Then m continues above M (resp. m and M are matched above, resp. M continues above m) if and only if there exist strings $s \subset m$ and $t \subset M$ such that s continues above t (resp. s and t are matched above, resp. t continues above s). A similar statement holds with the word "above" replaced by the word "below".*

Notation 39. For two sets of cells X and Y, let $\rightarrow_X (Y)$ (resp. $\uparrow_X (Y)$) denote the result of shifting to the right (resp. up), each row (resp. column) of Y by the number of cells of X in that row (resp. column). Define $\leftarrow_X (Y)$ and $\downarrow_X (Y)$ analogously.

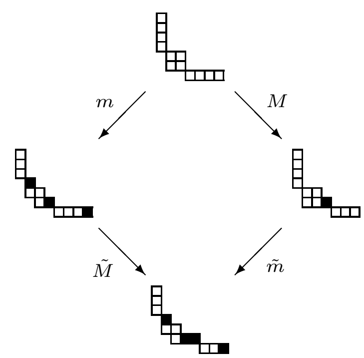
3.3. **Mixed elementary equivalence.**

Definition 40. A *mixed elementary equivalence* is a relation of the form (42) arising from a row move m and column move M from some $\lambda \in \Pi$, which has one of the following forms:

(1) m and M do not intersect and m and M are not contiguous. Then $\tilde{m} = m$ and $\tilde{M} = M$.
(2) m and M intersect and
  (a) m continues above and below M. Then
$$\tilde{m} = \rightarrow_M (m) \qquad \text{and} \qquad \tilde{M} = \rightarrow_m (M)$$
  (b) M continues above and below m. Then
$$\tilde{m} = \uparrow_M (m) \qquad \text{and} \qquad \tilde{M} = \uparrow_m (M).$$

Remark 41. If the pair (m, M) defines a mixed elementary equivalence then m and M are reasonable.

Example 42. For $k = 4$ the following diagram defines a mixed elementary equivalence via Case (2)(a). The black cells indicate those added to the original shape.



Proposition 43. *If (m, M) defines a mixed elementary equivalence, then the prescribed sets of cells $\tilde{m}$ and $\tilde{M}$ define a diamond equivalence.*

3.4. **Interfering row moves and perfections.** To define row equivalence we require the notions of interference and perfections.

Let m and M be row moves from $\lambda \in \Pi$ of respective ranks r and r' and lengths ℓ and ℓ' such that $m \cap M = \emptyset$.

Remark 44. Suppose a cell in the string s of m is above and contiguous with a cell in the string t of M. If all the cells of s are not above all the cells of t then using Property 26 and Lemma 35 one may deduce the contradiction that m and M intersect. If the cells of s are above those of t, we have a contradiction to Definition 17. Therefore m and M are not contiguous. In particular the diagrams of the strings of m and M are unaffected by the presence of the other move.

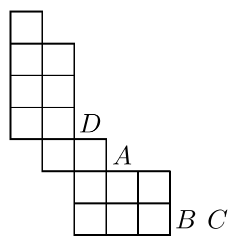
Say that the pair (m, M) is *interfering* if $m \cap M = \emptyset$ and $\mathrm{cs}(\lambda \cup m \cup M)$ is not a partition. Let $m = s_1 \cup \cdots \cup s_r$ and $M = t_1 \cup \cdots \cup t_{r'}$. We immediately have

Lemma 45. *Suppose (m, M) is interfering. Say the top cell of m is above the top cell of M. Then*
(1) $c^-_{s_1,d} = c_{t_{r'},u}$. *In particular m and M are nondegenerate.*
(2) *Every cell of m is above every cell of M.*
(3) $\mathrm{cs}(\lambda)_{c_{s_1},d} = \mathrm{cs}(\lambda)_{c_{t_{r'},u}} + 1$.

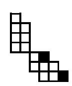
Property (1) tells us that the pair (m, M) can only be interfering if the last negatively modified column of M is just before the first positively modified column of m (or similarly with m and M interchanged).

Suppose (m, M) is interfering and the top cell of m is above the top cell of M. A *lower* (resp. upper) *perfection* of the pair (m, M) is a k-shape of the form $\lambda \cup m \cup M \cup m_{\text{per}}$ (resp. $\lambda \cup m \cup M \cup M_{\text{per}}$) where m_{per} (resp. M_{per}) is a $(\lambda \cup m \cup M)$-addable skew shape such that $m \cup m_{\text{per}}$ (resp. $M \cup M_{\text{per}}$) is a row move from $M * \lambda$ (resp. $m * \lambda$) of rank r (resp. r') and length $\ell + \ell'$ and $M \cup m_{\text{per}}$ (resp. $m \cup M_{\text{per}}$) is a row move from $m * \lambda$ (resp. $M * \lambda$) of rank $r + r'$ and length ℓ' (resp. ℓ). We say that (m, M) is lower-perfectible (resp. upper perfectible) if it admits a lower (resp. upper) perfection. By Lemma 47, the lower (resp. upper) perfection is unique if it exists.

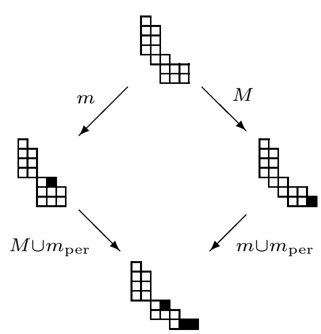
Example 46. For $k = 5$, row moves $m = \{A\}$ and $M = \{B\}$ from λ are pictured below together with $\partial\lambda$.

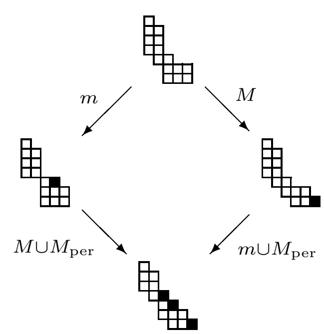


The pair (m, M) is interfering: the skew shape $\partial(\lambda \cup m \cup M)$ is pictured below.



The lower and upper perfections both exist, with $m_{\text{per}} = \{C\}$ and $M_{\text{per}} = \{D\}$. They are pictured as the bottom shapes in the left and right diagrams respectively.





Lemma 47. *Suppose (m, M) are interfering row moves with the top cell of m above that of M.*

(1) *Suppose a lower perfection ρ exists. Then it is unique: m_{per} is such that $m \cup m_{\text{per}}$ is the unique move from $\lambda \cup M$ obtained by extending each of the strings of m below by ℓ' cells, and also $M \cup m_{\text{per}}$ is the unique move from $\lambda \cup m$ obtained by adding r more translates to the right of the strings of M.*
(2) *Suppose an upper perfection ρ exists. Then it is unique: M_{per} is such that $M \cup M_{\text{per}}$ is the unique move from $\lambda \cup m$ obtained by extending each of the*

strings of M above by ℓ cells, and also $m \cup M_{\text{per}}$ is the unique move from $\lambda \cup M$ given by adding r' more translates to the left of the strings of m.

3.5. **Row elementary equivalence.**

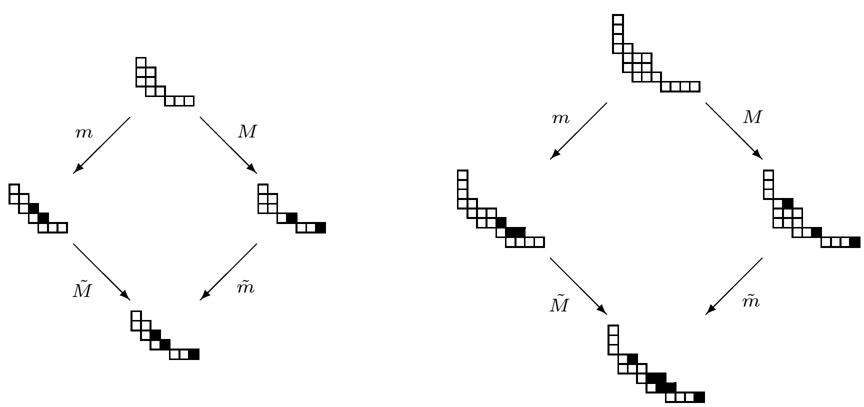
Definition 48. A *row elementary equivalence* is a relation of the form (42) arising from two row moves m and M from some $\lambda \in \Pi$, which has one of the following forms:

(1) m and M do not intersect and (m, M) is non-interfering. Then $\tilde{m} = m$ and $\tilde{M} = M$.
(2) (m, M) is interfering (and say the top cell of m is above the top cell of M) and (m, M) is lower (resp. upper) perfectible by adding the set of cells $x = m_{\text{per}}$ (resp. $x = M_{\text{per}}$). Then $\tilde{m} = m \cup x$ and $\tilde{M} = M \cup x$.
(3) m and M intersect and are matched above (resp. below). In this case $\tilde{m} = m \setminus (m \cap M)$ and $\tilde{M} = M \setminus (m \cap M)$.
(4) m and M intersect and m continues above and below M. In this case $\tilde{m} = \uparrow_{m \cap M}(m)$ and $\tilde{M} = \uparrow_{m \cap M}(M)$.
(5) $M = \emptyset$ and there is a row move m_{per} from $m * \lambda$ such that $m \cup m_{\text{per}}$ is a row move from λ. Then $\tilde{M} = m_{\text{per}}$ and $\tilde{m} = m \cup m_{\text{per}}$.

In cases (4) and (5) the roles of m and M may be exchanged. In case (2), (m, M) may be both lower and upper perfectible, in which case both perfections yield row elementary equivalences. In case (5), m_{per} can continue the strings of m above or below. This case can thus be considered as a degeneration of case (2).

Remark 49. If the pair (m, M) satisfies a row elementary equivalence then m and M are reasonable.

Example 50. We give examples of row elementary equivalences. For case (2) see Example 46. On the left we give for $k = 4$ a case (3) example with moves matched above, and on the right, for $k = 5$ an example of case (4).



Proposition 51. *If (m, M) defines a row elementary equivalence, then the prescribed sets of cells $\tilde{m}$ and $\tilde{M}$ define a diamond equivalence.*

3.6. **Column elementary equivalence.**

Definition 52. There is an obvious transpose analogue of row elementary equivalence which we shall call *column elementary equivalence.*

Since (44) is obviously satisfied in the case of a column elementary equivalence, the transpose analogue of Proposition 51 holds.

3.7. **Diamond equivalences are generated by elementary equivalences.**

Lemma 53. *Any diamond equivalence $\tilde{M}m \equiv \tilde{m}M$ in which m is a row move and M a column move from some $\lambda \in \Pi$, is a mixed elementary equivalence.*

Lemma 54. *Let m and M be row (resp. column) moves such that $\tilde{M}m \equiv \tilde{m}M$ is a diamond equivalence. Then the relation $\tilde{M}m \equiv \tilde{m}M$ can be generated by row (resp. column) elementary equivalences.*

We have immediately:

Proposition 55. *The equivalence relations generated respectively by diamond equivalences and by elementary equivalences are identical.*

Proof. Lemma 53 and Lemma 54 imply that diamond equivalences are generated by elementary equivalences. Since elementary equivalences are diamond equivalences by Propositions 43 and 51, the proposition follows. □

3.8. **Proving properties of mixed equivalence.**

For the rest of this subsection we assume that m and M are respectively row and column moves from $\lambda \in \Pi$.

Property 56. *Strings of m and M cannot be matched above or below.*

Proof. By Property 26(3) the cells of the strings of m and M that meet are λ-addable. The lemma then easily from considering the diagrams of m and M. □

Property 57. *Any string of m (resp. M) meets at most one string of M (resp. m).*

Proof. Suppose there is some string $s = \{a_1, a_2, \ldots\} \subset m$ where $a_i \in t$ and $a_j \in \bar{t}$ for distinct column-type strings $t, \bar{t} \in M$. Let i and j be such that $j - i$ is minimum where $i < j$.

We first show that a_i is not the bottom cell of t. If this were the case, the distance between the bottom cell of t and the bottom cell of $\bar{t}$ would be larger than $k-1$, which would contradict Lemma 24. Therefore a_i is not the lowest cell of t.

The cells a_i and a_j are λ-addable by Property 26(3) and so is s by Lemma 35. Thus by Remark 14, the cell of t contiguous to and below a_i is a_{i+1}. If $a_{i+1} \neq a_j$ we have a contradiction to the choice of i and j. If $a_{i+1} = a_j$ we have the contradiction that t and $\bar{t}$ intersect.

Taking transposes, every string of M meets at most one string of m. □

Lemma 58. *Suppose $s = \{a_1, \ldots, a_\ell\}$ and $t = \{b_1, \ldots, b_{\ell'}\}$ are strings in m and M respectively such that $s \cap t \neq \emptyset$. Then $s \cap t$ is a λ-addable string and there are intervals $A \subset [1, \ell]$ and $B \subset [1, \ell']$ such that $s \cap t = \{a_j \mid j \in A\} = \{b_j \mid j \in B\}$. Moreover, either $\min(A) = 1$ or $\min(B) = 1$, and also either $\max(A) = \ell$ or $\max(B) = \ell'$.*

Proof. Let $x \in s \cap t$. By Property 26 and Lemma 35 all the cells of s and t are λ-addable. Suppose both s and t contain cells below (resp. above) x. Since cells in strings satisfy a contiguity property, there are λ-addable cells $z \in s$ and $z' \in t$ such that z and z' are contiguous with and below (resp. above) x. By Remark 14, $z = z'$. The Lemma follows. □

Call a row-type (resp. column-type) string of m (resp. M) *primary* if it consists of λ-addable corners. Write $\mathrm{Prim}(m)$ for the set of primary strings of m; the dependence on λ is suppressed. The strings of m (resp. M) are totally ordered, and this induces an order on the primary strings. For $s \in \mathrm{Prim}(m)$ with $s \neq \max(\mathrm{Prim}(m))$ (resp. $s \neq \min(\mathrm{Prim}(m))$) we write $\mathrm{succ}(s)$ (resp. $\mathrm{pred}(s)$) for the cover (resp. cocover) of s in $\mathrm{Prim}(m)$.

Remark 59. By Lemma 58, if s is a string in m and t a string in M such that $s \cap t \neq \emptyset$ then $s \in \mathrm{Prim}(m)$ and $t \in \mathrm{Prim}(M)$.

Lemma 60. *Suppose s is a string in m and t a string in M such that $s \cap t \neq \emptyset$.*

(1) *If s continues below (resp. above) t and $s \neq \min(\mathrm{Prim}(m))$ (resp. $s \neq \max(\mathrm{Prim}(m))$) then there is a string $t' \in \mathrm{Prim}(M)$ such that $t' > t$ (resp. $t' < t$), $\mathrm{pred}(s) \cap t' \neq \emptyset$ (resp. $\mathrm{succ}(s) \cap t' \neq \emptyset$), and $\mathrm{pred}(s)$ (resp. $\mathrm{succ}(s)$) continues below (resp. above) t'.*

(2) *If t continues above (resp. below) s and $t \neq \min(\mathrm{Prim}(M))$ (resp. $t \neq \max(\mathrm{Prim}(M))$), then there is a string $s' \in \mathrm{Prim}(m)$ such that $s' > s$ (resp. $s' < s$), $s' \cap \mathrm{pred}(t) \neq \emptyset$ (resp. $s' \cap \mathrm{succ}(t) \neq \emptyset$), and $\mathrm{pred}(t)$ (resp. $\mathrm{succ}(t)$) continues above (resp. below) s'.*

Proof. We prove (1) as (2) is the transpose analogue. Suppose s continues below t and $s \neq \min(\mathrm{Prim}(m))$. Let b be the bottom cell in t; it is also the bottom cell of $s \cap t$. By hypothesis the string s has a λ-addable cell $b' \notin t$, contiguous to and below b. M shortens the row of b' since $(\mathrm{row}(b'), \mathrm{col}(b))$ is removed by M and b' is not added by M by Property 57. Let c and c' be the translates in $\mathrm{pred}(s)$ of the cells b and b' in s. Note that $\mathrm{row}(b') < \mathrm{row}(c')$ since b' and c' are λ-addable and $c' \in \mathrm{pred}(s)$. Furthermore, by Corollary 34, $\mathrm{rs}(\lambda)_{\mathrm{row}(c')} = \mathrm{rs}(\lambda)_{\mathrm{row}(b')}$. Now, from a previous comment $\mathrm{rs}(M * \lambda)_{\mathrm{row}(b')} = \mathrm{rs}(\lambda)_{\mathrm{row}(b')} - 1$. In order for $M * \lambda$ to belong to Π, M must also remove the cell $(\mathrm{row}(c'), \mathrm{col}(c))$ without adding the cell c'. Therefore there is a string $t' > t$ such that $\mathrm{pred}(s) \cap t' \neq \emptyset$ and such that $\mathrm{pred}(s)$ continues below t'.

Suppose s continues above t and $s \neq \max(\mathrm{Prim}(m))$. Let b be the highest cell in t, b' the cell of s below and contiguous with b. Let c and c' be the translates in $\mathrm{succ}(s)$ of b and b' in s. One may show that M adds a cell to the row of b and removes none. M must do the same to the row of c since $M * \lambda \in \Pi$. The rest of the argument is similar to the previous case. $\square$

Lemma 61. *Lemma 38 holds for a row move and a column move.*

Proof. Suppose s and s' are strings of m and t and t' are strings of M such that $s \cap t \neq \emptyset$, $s' \cap t' \neq \emptyset$, s continues below t, and s' does not continue below t'. By Lemma 60, $s' > s$. Let b, b', c, c' be the bottom cells of s, s', t, t'. We have that $d(c) < d(b) < d(b') \leq d(c')$. Since the distance between c and b is more than $k - 1$, the distance between t and t' is also more than $k - 1$. But this violates Lemma 24. $\square$

Proof of Proposition 43. Let $m = \{s_1, s_2, \ldots, s_r\}$ and $s_1 = \{a_1, \ldots, a_\ell\}$.

(1) The disjointness of m and M implies the commutation of (43), and (44) holds trivially in this case. So it suffices to show that m is a row move from $M * \lambda$; showing that M is a column move from $m * \lambda$ is similar.

Since $M \cap m = \emptyset$, s_1 is a $(M * \lambda)$-addable string. The diagram of the string s_1 remains the same in passing from λ to $M * \lambda$; the only place it could change is

in the row of a_1 and the column of a_ℓ, and this could only occur if a_1 or a_ℓ were contiguous with a cell of M, which is false by assumption. So s_1 is a row-type $(M * \lambda)$-addable string. The argument for the other strings of m is similar.

Since M is a column move from λ, $\mathrm{cs}(\lambda) = \mathrm{cs}(M * \lambda)$. But then Property 22 implies that $(M * \lambda) \cup m \in \Pi$. This proves that m is a row move from $M * \lambda$ as required.

(2) We prove case (a) as (b) is similar. By definition of $\tilde{M}$, $\tilde{M}$ contains the same number of strings as M and the strings of $\tilde{M}$ are of the same length as those of M. Thus (44) is satisfied.

By Lemma 60, all primary strings of m meet M. In particular, since the first string s_1 of m is always primary, it meets M. The string s_1 meets a single string $\hat{s}$ in M and $s_1 \cap \hat{s} = \hat{s} = \{a_p, a_{p+1}, \ldots, a_n\}$ for some $1 < p \leq n < \ell$ by Lemma 58 since s_1 continues above and below $\hat{s}$.

We now show that $t = \rightarrow_M (s_1)$ is a $(M * \lambda)$-addable string. By Property 26 M is a vertical strip and

$$t = \{a_1, a_2, \ldots, a_p^\dagger, a_{p+1}^\dagger, \ldots, a_n^\dagger, a_{n+1}, \ldots, a_\ell\}$$

where $a^\dagger$ denotes the cell right-adjacent to the cell a.

Since a_p is the top cell of the column-type λ-addable string $\hat{s}$ and there is a λ-addable corner a_{p-1} contiguous to and above a_p, by Definition 17 we have $d(a_p) - d(a_{p-1}) = k$ and $d(a_p^\dagger) - d(a_{p-1}) = k+1$. Similarly, since a_n is the bottom cell of $\hat{s}$, we have $d(a_{n+1} - d(a_n) = k+1$ and $d(a_{n+1}) - d(a_n^\dagger) = k$. Thus the cells of t satisfy the contiguity conditions for a string.

Let $\mu = M * \lambda$. For $i < p$ and $i > n$, a_i is μ-addable since it is λ-addable and $a_i \notin M$ by Property 57.

Let c be the column of a_{p-1}. Since a_p is the top cell of a string in M, there is no cell removed in the row of a_p when going from λ to μ, and thus $\mathrm{bot}_c(\partial\mu)$ still lies in the row of a_p. By Remark 13 there is a μ-addable corner in the row of a_p and it corresponds to $a_p^\dagger$.

Now observe that if $a_{p+1}^\dagger$ is not a μ-addable corner, then there is a μ-addable corner e below it by Lemma 33 (since $a_{p+1}^\dagger$ is a distance k or $k+1$ from the μ-addable corner $a_p^\dagger$). Since M is a column move, there is a μ-removable string with cells c_p and c_{p+1} in the columns of a_p and a_{p+1} which is a translate of string $\hat{s}$ (the two strings may coincide). The distance between $a_p^\dagger$ and e is thus larger (by exactly one unit) than the distance between c_p and c_{p+1}. Furthermore, $a_p^\dagger$ and e lie in columns immediately to the right of those of c_p and c_{p+1} respectively. We then have the contradiction that the removable string containing c_p and c_{p+1} and the addable string containing $a_p^\dagger$ and e violate Lemma 32. Therefore $a_{p+1}^\dagger$ is a μ-addable corner and repeating the previous argument again and again we get that $a_i^\dagger$ is μ-addable for any $p < i \leq n$.

Therefore t is a μ-addable string. It is of row-type since the top and bottom of its string diagram are unaffected by adding M to λ and coincide with the top and bottom of the diagram of the λ-addable row-type string s.

Suppose there are q strings of m in the rows of s_1, and let $t_j = \rightarrow_M (s_j)$ for $1 \leq j \leq q$. It follows from the results of Subsection 3.8 and the translation property of strings in row moves, that t_j is a translate of t_1: the top and bottom of t_j agree with those of s_j, and $\rightarrow_M$ right-shifts the p-th through n-th cells in s_j to obtain t_j,

which are the same positions within the string s_1 that are right-shifted to obtain t_1. In particular t_j is a row-type string.

We claim that t_j is $(M \cup t_1 \cup \cdots \cup t_{j-1}) * \lambda$-addable for $1 \leq j \leq q$. It holds for $j = 1$. For the general case, since m is a horizontal strip, we have that $\lambda_{\mathrm{row}(a_{p-1})-1} - \lambda_{\mathrm{row}(a_{p-1})} \geq q$. In μ we still have $\mu_{\mathrm{row}(a_{p-1})-1} - \mu_{\mathrm{row}(a_{p-1})} \geq q$ since there is no cell of M in $\mathrm{row}(a_p)$. By Lemma 31 applied to the string t we have that $\mu_{\mathrm{row}(a_i)-1} - \mu_{\mathrm{row}(a_i)} \geq q$ for any $p \leq i \leq n$. This immediately implies that t_j is $(M \cup t_1 \cup \cdots \cup t_{j-1}) * \lambda$-addable for $1 \leq j \leq q$.

The same approach shows that $t'_i = \rightarrow_M (s_i)$ is a μ-addable string for any other primary string s_i of m, and that the strings lying in the rows of s_i can be right-shifted as prescribed. Moreover, arguing as in the proof of Lemma 61, one may show that $\hat{s}_i = s_i \cap t_i$ consists of the p-th through n-th cells of s_i (using the same p and n as for s_1). It follows that all the strings $\tilde{s}_i$ are translates of each other.

$\tilde{m} * M * \lambda$ is a k-shape, because $M * \lambda$ is, and because the condition in Property 22 is unchanged in passing from the move $\lambda \to m * \lambda$, to the move $M * \lambda \to \tilde{m} * M * \lambda$. Therefore $\tilde{m}$ is a row move from $M * \lambda$ with first strings $\tilde{s}_1, \ldots, \tilde{s}_r$.

We must show that $\tilde{M}$ is a column move from $m * \lambda$. It is a vertical strip, being the difference of partitions $m * \lambda$ and $\lambda \cup M \cup \tilde{m}$, and having at most one cell per row by definition.

It was shown previously that for any string s' of M that meets m, s' is contained in the string of m that it meets. Note that since strings in a move are translates of each other, we have that if the primary string $t = \{a_1, \ldots, a_\ell\}$ of m is such that there are n cells of m in the row of a_1, then there are also n cells of m in the row of a_i for all i. It follows that under $\rightarrow_m$, every string in M is translated directly to the right by some number of cells (possibly zero). Therefore $\tilde{M}$ is the disjoint union of strings that are translates of each other and which start in consecutive rows. Since $\tilde{M}$ is an $m * \lambda$-addable vertical strip we deduce that it is a column move from $m * \lambda$. □

3.9. **Proving properties of row equivalence.** We state the analogues of results in Subsection 3.8 for intersections of row moves m and M from $\lambda \in \Pi$.

Property 62. *Every string of m meets at most one string of M.*

Lemma 63. *Suppose $s = \{a_1, \ldots, a_\ell\}$ and $t = \{b_1, \ldots, b_{\ell'}\}$ are strings in m and M respectively such that $s \cap t \neq \emptyset$. Then $s \cap t$ is a string and there are intervals $A \subset [1, \ell]$ and $B \subset [1, \ell']$ such that $s \cap t = \{a_j \mid j \in A\} = \{b_j \mid j \in B\}$. Moreover, either $\min(A) = 1$ or $\min(B) = 1$, and also either $\max(A) = \ell$ or $\max(B) = \ell'$.*

Lemma 64. *Suppose $m = s_1 \cup s_2 \cup \cdots \cup s_p$ and $M = t_1 \cup t_2 \cup \cdots \cup t_q$ are row moves on $\lambda \in \Pi$ with given string decomposition such that $m \cap M \neq \emptyset$.*

(1) *The leftmost cell of $m \cap M$ is contained in either s_1 or t_1.*
(2) *The rightmost cell of $m \cap M$ is contained in either s_p or t_q.*

Lemma 65. *Lemma 38 holds for m and M both row moves.*

Lemma 66. *Suppose $m = s_1 \cup s_2 \cup \cdots \cup s_p$ and $M = t_1 \cup t_2 \cup \cdots \cup t_q$ are intersecting moves from $\lambda \in \Pi$.*

(1) *Suppose that m continues above M but the two are matched below. Then $p \leq q$ and s_i contains t_i and continues above it for $1 \leq i \leq p$.*
(2) *Suppose that m continues below M but the two are matched above. Then $p \leq q$ and s_{p-i} contains t_{q-i} and continues below it for $0 \leq i < p$.*

Proof. We prove (1) as (2) is similar. The hypotheses imply that for some i and j we have $c_{s_i,d} = c_{t_j,d}$. It follows from Property 21 that $c_{s_1,d} = c_{t_1,d}$, that is, s_1 and t_1 intersect. Applying Property 21 to the upper part of M we conclude that $p \le q$. We have that s_i meets t_i for $1 \le i \le p$ since it is true for $i = 1$ and strings in a move are translates. Since m continues above M and they are matched below, s_i contains t_i and continues above it. □

Proof of Lemma 47. We prove (1) as (2) is similar. Let m_{per} give rise to the lower perfection $\lambda \cup m \cup M \cup m_{\mathrm{per}} \in \Pi$. Since $m \cup m_{\mathrm{per}}$ is a row move from $M * \lambda$ of rank r, it follows that m_{per}, viewed as $\lambda \cup m \cup M$-addable, must negatively modify (by -1) precisely the columns $c_{s_1,u}$ through $c_{s_r,u}$. So $M \cup m_{\mathrm{per}}$ is a row move from $m * \lambda$ which negatively modifies the $r + r'$ consecutive columns $c_{t_1,u}, \dots, c_{t_{r'},u}, c_{s_1,d}, \dots, c_{s_r,d}$. Therefore m_{per} is specified by adjoining to $\lambda \cup m \cup M$, translates of t_1 in the r columns just after $c_{t_{r'},u}$. The other claims are clear. □

Proof of Proposition 51. Cases (1) and (4) are similar to Cases (1) and (2) of mixed equivalence. Case (5) is trivial. Case (2) holds by definition. So consider Case (3). We suppose that m and M are row moves on λ that are matched below, as the "matched above" case is similar. If m and M are also matched above then it follows that $m = M$: intersecting strings must coincide, and Property 21 implies that the two moves must modify the same columns. So we may assume that m continues above M. Using the notation of Lemma 66, we see that $\tilde{M}$ decomposes into strings $t_{p+1}, \dots, t_q$. These strings neither intersect nor have any cells contiguous with any of the other strings in m or M. It follows that $\tilde{M}$ is a row move from $m * \lambda$ since $\mathrm{cs}(\tilde{M} \cup m \cup \lambda) = \mathrm{cs}(M \cup m \cup \lambda)$ is a partition (m and M are not interfering). As a set of cells, $\tilde{m}$ decomposes into strings $u_i := s_i \setminus t_i$ for $1 \le i \le p$ that are translates of each other. The top of the diagram of the string u_1 coincides with that of s_1. Consider the column $c_{u_1,d}$ in the diagram of u_1 as a $M * \lambda$-addable string. u_1 does not remove a cell from this column since the first string t_1 of M already removed such a cell in passing from $\partial\lambda$ to $\partial M * \lambda$. Therefore u_1 is a row-type $M * \lambda$-addable string. Similarly it follows that $\tilde{m}$ is a row move from $M * \lambda$ with strings u_i. □

3.10. **Proofs of Lemma 53 and Lemma 54.**

Proof of Lemma 53. By Definition 40 we need to show that if m and M do not intersect but a cell of m is contiguous to a cell of M or if m and M intersect and are not reasonable, then m and M do not define a diamond equivalence.

Suppose there is a diamond equivalence $\tilde{m}M \equiv \tilde{M}m$. By definition we must have $\Delta_{\mathrm{rs}}(\tilde{M}) = \Delta_{\mathrm{rs}}(M)$ (m and $\tilde{m}$ are row moves and thus do not change row shapes). As a consequence, M and $\tilde{M}$ must have the same rank, and similarly for m and $\tilde{m}$. The charge conservation of a diamond equivalence also implies that M and $\tilde{M}$ have the same length.

Consider the case where m and M do not intersect but a cell of m is contiguous to a cell of M. Suppose m is above M. Then the bottom cell a of a given string s of m is contiguous to the top cell b of a given string t of M. Furthermore, $d(b) - d(a) = k$, for otherwise s and t would not be of row and column types respectively. Since $\Delta_{\mathrm{rs}}(t)$ has a $+1$ in $\mathrm{row}(b)$, there must be a string $\tilde{t}$ of $\tilde{M}$ that ends in $\mathrm{row}(b)$ in order for $\Delta_{\mathrm{rs}}(\tilde{M})$ to have a $+1$ in $\mathrm{row}(b)$. But then in $m * \lambda$ the hook-length of the cell in position $(\mathrm{row}(b), \mathrm{col}(a))$ is k which gives the contradiction that $\tilde{t}$ is not a column-type string. Otherwise m is below M, the top cell a of some string $s \subset m$

is contiguous with the bottom cell b of some string $t \subset M$ with $d(a) - d(b) = k+1$. Let $r = \mathrm{row}(a)$ and $c = \mathrm{col}(b)$. Then $h_\lambda(r,c) = k$. Since $\Delta_{\mathrm{cs}}(m) = \Delta_{\mathrm{cs}}(\tilde{m})$, $\tilde{m}$ must remove $(r',c) = \mathrm{bot}_c(\partial(M * \lambda))$ where $r' > r$. Since M is a column move, $\mathrm{cs}(M*\lambda)_c = \mathrm{cs}(\lambda)_c$. Since m contains the cell $a = (r, \lambda_r + 1)$, $\tilde{m}$ must contain the cell $(r', \lambda_r + 1)$ in order to remove (r', c). But then M contains the cell a, contradicting the disjointness of m and M.

Now consider the case where m and M intersect but are not reasonable. Suppose there is a string s of m that meets a string t of M, with s continuing below t but not above it. By Property 56, we know that t finishes above s. Let $t = \{a_1, \ldots, a_\ell\}$ and s be such that $s \cap t = \{a_i, \ldots, a_\ell\}$, and let b be the cell of s contiguous to and below a_ℓ (it exists by our hypotheses). Since M is a column move, $\Delta_{\mathrm{rs}}(M)$ has a -1 in $\mathrm{row}(b)$. Thus $\Delta_{\mathrm{rs}}(\tilde{M})$ must also have a -1 in $\mathrm{row}(b)$. This implies that there is a string $t' = \{a'_1, \ldots, a'_\ell\}$ of $\tilde{M}$ (recall that M and $\tilde{M}$ have the same length) such that $\Delta_{\mathrm{rs}}(t')$ has a -1 in $\mathrm{row}(b)$. By definition of column moves, and since $\Delta_{\mathrm{rs}}(M) = \Delta_{\mathrm{rs}}(\tilde{M})$ (which implies that M and $\tilde{M}$ have the same rank), we have that the upper cells of t and t' must coincide. That is, $\{a'_1, \ldots, a'_{i-1}\} = \{a_1, \ldots, a_{i-1}\}$. Note that since m is a horizontal strip and $\tilde{M}$ is a vertical strip, the cells outside $m*\lambda$ catty-corner to $\{a_i, \ldots, a_\ell\}$ are not in $\tilde{M}m*\lambda$. Now, the distance between a_{i-1} and a_i is $k+1$ (a_i is the top cell of a row move). Thus from the previous comment and contiguity we have that $d(a'_i) < d(a_i) < d(a'_{i+1}) < d(a_{i+1}) < \cdots < d(a'_\ell) < d(a_\ell)$. But then we have the contradiction that a'_ℓ cannot negatively modify $\mathrm{row}(b)$ since in this row there is no cell of $\partial(m*\lambda)$ weakly to the left of $\mathrm{col}(a_\ell)$. The case where there is a string s of m that meets a string t of M, with s continuing above t but not below it is similar. □

Proof of Lemma 54. All cases that could produce a diamond equivalence where m and M do not intersect are covered by Definition 48. In case (1) there are no strings that could be added at the same time to m and M to produce moves $\tilde{m} \neq m$ and $\tilde{M} \neq M$. In case (2), unicity is guaranteed by Lemma 47.

Suppose we have a diamond equivalence $\bar{M}m \equiv \bar{m}M$, where m and M are such as in case (3), and suppose that and m and M are matched below with m continuing above. As mentioned in the proof of Proposition 51, $\tilde{m}$ decomposes into strings $u_i = s_i \setminus t_i$ for $1 \leq i \leq p$ and $\tilde{M}$ decomposes into strings $t_{p+1}, \ldots, t_q$. We now show that if $q > p$ then $\bar{m} = \tilde{m}$ and $\bar{M} = \tilde{M}$. It is obvious that $\tilde{m} \subseteq \bar{m}$ and $\tilde{M} \subseteq \bar{M}$. There are two possible options: either $\bar{M}$ has more strings than $\tilde{M}$ or its strings are extensions of those of $\tilde{M}$. Since $\bar{m} \setminus \tilde{m} = \bar{M} \setminus \tilde{M}$, in the first option the extra strings must extend the u_i's below, and in the second option the extension must form strings to the right of those of $\tilde{m}$. The former is impossible since the distance between the bottom cell of any u_i and the top cell of any of the new strings added is more than $k+1$. The latter case is impossible since no new strings can be added to the right of $\tilde{m}$ to form a move by Property 21. Thus the only option is $p = q$.

In this case $\tilde{M} = \emptyset$, $\tilde{m} = m \setminus M$ and we have:

$$\begin{tikzcd}
& \cdot \arrow[dl, "M"'] \arrow[dr, "m"] & \\
\cdot \arrow[dr, "m\backslash M"] \arrow[ddr, "\bar{m}"'] & & \cdot \arrow[dl, "\emptyset"'] \arrow[ddl, "\bar{M}"] \\
& \cdot \arrow[d, "\bar{M}"] & \\
& \cdot &
\end{tikzcd} \tag{47}$$

New strings cannot be added to $m \setminus M$ to form a new move. Strings to the right would violate Property 21. And strings to the left need to be such that in M the columns $c_{t_1,u}$ and the one to its left are of the same size (and thus M could not have been a move). So $m \setminus M$ can be extended either below or above (not both since otherwise $\bar{M}$ could not be a move). In this case the triangle in the left of the diagram obeys a relation of the form (3). Since the other triangle is trivial (a case (5) with $m_{\rm per} = \emptyset$), the diamond equivalence $\bar{M}m = \bar{m}M$ is generated by the elementary ones. The case (3) where m and M are matched above is similar.

The only other cases that could produce a diamond equivalence which are not covered by Definition 48 are those where m and M are not reasonable, that is, there are strings s and t of m and M respectively such that $s \cap t \neq \emptyset$, $t \not\subseteq s$ and $s \not\subseteq t$. Suppose that t continues below s. We show that if there are strings $t_i, \ldots, t_{i+j}$ of M that do not intersect strings of m then there is no possible diamond equivalence $\bar{M}m = \bar{m}M$. The strings $t_i, \ldots, t_{i+j}$ need to be to the right of those that meet strings of m by Property 21 applied to the positively modified columns of m. For the diamond equivalence to hold, we need a $M_{\rm per}$ that extends the strings $t_i, \ldots, t_{i+j}$ above and that add extra strings to the right of $m \setminus M$. But this is impossible by Property 21. In a similar way, if there are strings of m that do not intersect strings of M then there is no possible diamond equivalence. Therefore, we are left with the case where the strings $s_1, \ldots, s_p$ of m and $t_1, \ldots, t_p$ of M each intersect with one another. In this case we necessarily have $\bar{m} = m \setminus M$ and $\bar{M} = M \setminus m$. But then $\bar{m}M = \bar{M}m = \mathcal{M}$ is also a move and we have the situation.

$$\begin{tikzcd}
& \cdot \arrow[dl, "m"'] \arrow[dr, "M"] \arrow[dd, "\mathcal{M}"] & \\
\cdot \arrow[dr, "M\backslash m"'] & & \cdot \arrow[dl, "m\backslash M"] \\
& \cdot &
\end{tikzcd} \tag{48}$$

In this case both triangles correspond to Case (3) of Definition 48 and thus this diamond equivalence is also generated by elementary ones. $\square$

4. Strips and tableaux for k-shapes

In this section we introduce a notion of (horizontal) strip and tableau for k-shapes.

4.1. **Strips for cores.**

We recall from [11, 8] the notion of weak strip and weak tableau for cores. Let $\tilde{S}_{k+1}$ and S_{k+1} be the affine and finite symmetric groups and let $\tilde{S}^0_{k+1}$ denote the set of minimal length coset representatives for $\tilde{S}_{k+1}/S_{k+1}$. $\mathcal{C}^{k+1}$ has a poset structure given by the left weak Bruhat order transported across the bijection $\tilde{S}^0_{k+1} \to \mathcal{C}^{k+1}$. Explicitly, μ covers λ in $\mathcal{C}^{k+1}$ if μ/λ is a nonempty maximal λ-addable string. Such a string is always of cover-type and consists of all λ-addable cells whose diagonal indices have a fixed residue (say i) mod $k+1$, and corresponds to a length-increasing left multiplication by the simple reflection $s_i \in \tilde{S}_{k+1}$. A *weak strip* in $\mathcal{C}^{k+1}$ is an interval in the left weak order whose corresponding skew shape is a horizontal strip; its *rank* is the height of this interval, which coincides with the number of distinct residues mod $k+1$ of the diagonal indices of the cells of the corresponding skew shape. For $\lambda \subset \mu$ in $\mathcal{C}^{k+1}$, a *weak tableau* T of shape μ/λ is a chain

$$\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(N)} = \lambda$$

in $\mathcal{C}^{k+1}$ where each interval $\lambda^{(i-1)} \subset \lambda^{(i)}$ is a weak strip. The weight of a weak tableau T is the sequence of nonnegative integers $\mathrm{wt}(T)$ whose i-th member $\mathrm{wt}(T)_i$ is the rank of $\lambda^{(i)}/\lambda^{(i-1)}$. Let $\mathrm{WTab}^{k+1}_{\mu/\lambda}$ be the set of weak tableaux of $k+1$-cores of shape μ/λ. The weight generating function of $\mathrm{WTab}^{k+1}_{\mu/\lambda}$ is denoted by $\mathrm{Weak}^{(k+1)}_{\mu/\lambda}[X]$.

4.2. **Strips for k-shapes.**

Definition 67. A strip of rank r is a horizontal strip μ/λ of k-shapes such that $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip and $\mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$ is a vertical strip, both of size r. A cover is a strip of rank 1.

By the assumption that $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip, distinct modified rows of μ/λ do not have the same length (in either $\mathrm{rs}(\lambda)$ or $\mathrm{rs}(\mu)$). The modified columns however form *groups* which have the same length in both $\mathrm{cs}(\mu)$ and $\mathrm{cs}(\lambda)$, where by definition two modified columns c, c' are in the same group if and only if $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c'}$.

Proposition 68. *A strip $S = \mu/\lambda$ has rank at most k.*

Proof. Suppose μ/λ has rank greater than k, that is, $|\mathrm{cs}(\mu)| - |\mathrm{cs}(\lambda)| > k$. Since $\mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$ is a horizontal strip, its cells occur in different columns. Therefore the k-bounded partition $\mathrm{cs}(\mu)$ has more than k columns, a contradiction. □

Remark 69. Although strips of rank k exist, in the remainder of the article **we shall only admit strips of rank strictly smaller than** k. For the purposes of this paper, this restriction is not so important: in Theorem 4, mod the ideal I_{k-1}, monomials with a multiple of x_i^k are killed, and therefore we choose to leave such tableaux out of the generating function by definition. Remark 76 will further elaborate on the effects of allowing strips of rank k in our construction.

The notion of a strip generalizes that of weak strips for k-cores and $k+1$-cores.

Proposition 70. *Suppose μ/λ is a strip such that $\mu,\lambda \in \mathcal{C}^{k+1}$ (resp. $\mu,\lambda \in \mathcal{C}^k$). Then μ/λ is a weak strip in $\mathcal{C}^{k+1}$ (resp. $\mathcal{C}^k$).*

Proof. It was established in [11] that if $\mu,\lambda \in \mathcal{C}^{k+1}$, $\mathrm{rs}(\lambda)/\mathrm{rs}(\mu)$ is a horizontal strip and $\mathrm{cs}(\lambda)/\mathrm{cs}(\mu)$ is a vertical strip, then μ/λ is a horizontal strip (Proposition 54 of [11]) and the cells in μ/λ correspond to one letter in a k-tableau (Theorem 71 of [11]). It was further established in Lemma 9.1 of [8] that k-tableaux and weak tableaux (sequences of weak strips in $\mathcal{C}^{k+1}$) are identical. Therefore λ/μ is a weak strip in $\mathcal{C}^{k+1}$. The same argument works for $\mu,\lambda \in \mathcal{C}^k$. □

4.3. Maximal strips and tableaux.

Definition 71. Let $\lambda \in \Pi$ be fixed. Let $\mathrm{Strip}_\lambda \subset \Pi$ be the induced subgraph of $\nu \in \Pi$ such that ν/λ is a strip. Moves (paths) in Strip_λ are called λ-augmentation moves (paths). By abuse of language, if m is a move (path) from μ to ν in Strip_λ we shall say that m is a λ-augmentation move (path) from the strip μ/λ to the strip ν/λ. An augmentation of a strip $S = \mu/\lambda$ is a strip reachable from S via a λ-augmentation path. A strip $S = \mu/\lambda$ is maximal if it is maximal in Strip_λ, that is, if it admits no λ-augmentation move.

Diagrammatically, an augmentation move is such that the following diagram commutes

$$\begin{tikzcd} \lambda \arrow[r, "\emptyset"] \arrow[d, "S"'] & \lambda \arrow[d, "\tilde{S}"] \\ \mu \arrow[r, "m"'] & \nu \end{tikzcd}$$

where S and $\tilde{S}$ are strips and $\emptyset$ denotes the empty move.

These definitions depend on a fixed $\lambda \in \Pi$, which shall usually be suppressed in the notation. Later we shall consider augmentations of a given strip S, meaning λ-augmentations where $S = \mu/\lambda$.

Clearly augmentation paths pass through strips of a constant rank.

Definition 72. Let $\mu \in \Pi$ be fixed. Let $\mathrm{Strip}^\mu \subset \Pi$ be the induced subgraph of $\rho \in \Pi$ such that μ/ρ is a strip. A strip $S = \mu/\rho$ is reverse-maximal if ρ is minimal in the graph Strip^μ (see Definition 168 for more details).

Let $\mu \supset \lambda$ with $\lambda,\mu \in \Pi$. A *(k-shape) tableau* of shape μ/λ is a sequence $\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)} = \mu$ with $\lambda^{(i)} \in \Pi$, such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a strip for all i. It is *maximal* (resp. *reverse-maximal*) if its strips are. The tableau has weight $\mathrm{wt}(T) = (a_1, a_2, \ldots, a_N)$ where a_i is the rank of the strip $\lambda^{(i)}/\lambda^{(i-1)}$ (which we require to be strictly smaller than k by Remark 69). Let

$$\widetilde{\mathfrak{S}}^{(k-1)}_{\mu/\lambda}[X] = \sum_{T \in \widetilde{\mathrm{Tab}}^k_{\mu/\lambda}} x^{\mathrm{wt}(T)} \tag{49}$$

$$\mathfrak{S}^{(k)}_{\mu/\lambda}[X] = \sum_{T \in \mathrm{Tab}^k_{\mu/\lambda}} x^{\mathrm{wt}(T)}. \tag{50}$$

where $\widetilde{\mathrm{Tab}}^k_{\mu/\lambda}$ (resp. $\mathrm{Tab}^k_{\mu/\lambda}$) denotes the set of maximal (resp. reverse-maximal) tableaux of shape μ/λ for $\lambda,\mu \in \Pi^k$.

For k-cores (resp. $k+1$-cores), the maximal (resp. reverse-maximal) tableau generating functions reduce to dual $k-1$ (resp. k) Schur functions. The following result is a consequence of Propositions 106 and 171.

Proposition 73.

(1) *For any $\lambda \in \mathcal{C}^k$ and $\mu \in \Pi^k$ such that μ/λ is a maximal strip, $\mu \in \mathcal{C}^k$. In particular, for $\lambda \in \Pi^k$, $\widetilde{\mathrm{Tab}}^k_\lambda$ is empty unless $\lambda \in \mathcal{C}^k$ and in that case $\widetilde{\mathrm{Tab}}^k_\lambda = \mathrm{WTab}^k_\lambda$ and the definition* (49) *of $\widetilde{\mathfrak{S}}^{(k-1)}_\lambda[X]$ agrees with the usual definition of the dual $(k-1)$-Schur function (or affine Schur function or weak Schur function) $\mathrm{Weak}^{(k)}_\lambda[X]$ via weak tableaux.*

(2) *For any $\mu \in \mathcal{C}^{k+1}$ and $\lambda \in \Pi^k$ such that μ/λ is a reverse-maximal strip, $\lambda \in \mathcal{C}^{k+1}$. In particular, for $\lambda \in \mathcal{C}^{k+1}$ and for every weight $\beta = (\beta_1, \beta_2, \dots)$ with $\beta_i \le k-1$ for all i, the set of reverse maximal tableaux of shape λ and weight β is equal to the set of weak k-tableaux of shape λ and weight β and thus $\mathfrak{S}^{(k)}_\lambda[X] = \mathrm{Weak}^{(k)}_\lambda[X] \mod I_{k-1}$ where I_{k-1} is defined in* (10).

Corollary 74. *For $\lambda \in \mathcal{C}^k$, we have $\widetilde{\mathfrak{S}}^{(k-1)}_\lambda[X] = \mathfrak{S}^{(k-1)}_\lambda[X]$.*

Theorem 4 is established as follows.

Theorem 75. *For all fixed $\mu, \nu \in \Pi$, there is a bijection*

$$\bigsqcup_{\lambda \in \Pi} \left(\mathrm{Tab}_{\mu/\lambda} \times \overline{\mathcal{P}}^k(\lambda, \nu) \right) \to \bigsqcup_{\rho \in \Pi} \left(\widetilde{\mathrm{Tab}}_{\rho/\nu} \times \overline{\mathcal{P}}^k(\mu, \rho) \right)$$

$$(S, [\mathbf{p}]) \mapsto (T, [\mathbf{q}]) \tag{51}$$

such that

$$\mathrm{wt}(S) = \mathrm{wt}(T).$$

The map $(S, [\mathbf{p}]) \to (T, [\mathbf{q}])$ is called the *pushout* and the inverse bijection $(T, [\mathbf{q}]) \to (S, [\mathbf{p}])$ is called the *pullback*, in reminiscence of homological diagrams, as the following diagram "commutes" for some λ, ρ:

$$\begin{tikzcd} \lambda \arrow[r, "{[\mathbf{p}]}"] \arrow[d, "S"'] & \nu \arrow[d, "T"] \\ \mu \arrow[r, "{[\mathbf{q}]}"'] & \rho \end{tikzcd}$$

Since tableaux are sequences of strips, we can immediately reduce the pushout bijection to the case that S and T are both single strips. One might try to straightforwardly reduce to the case that paths $\mathbf{p}$ and $\mathbf{q}$ are single moves m and m'. This does not work: not all pairs (S, m) admit a pushout. Those that do will be called *compatible*. The bijection (51) is defined by combining certain moves (called augmentation moves) with pushouts of compatible pairs. The proof of Theorem 75 will be completed in §15.

Proof of Theorem 4. Let ν be the empty k-shape. Then the only possibility for λ is the empty k-shape, S runs over Tab^k_μ, and by Proposition 73, ρ runs over $\mathcal{C}^k$ and T over WTab^k_ρ. By Theorem 75, we thus have a bijection between Tab^k_μ and $\bigsqcup_{\rho \in \mathcal{C}^k} \left(\mathrm{WTab}^k_\rho \times \overline{\mathcal{P}}^k(\mu, \rho) \right)$. Theorem 4 follows since it is known that each generating function $\mathrm{Weak}^{(k-1)}_\rho[X]$ is a symmetric function [12]. $\square$

Proof of Theorem 5. The k-Schur functions satisfy (essentially by definition) the Pieri rule [12] $h_r[X]\, s^{(k)}_\mu[X] = \sum_\rho s^{(k)}_\rho[X]$ where the sum is over weak strips ρ/μ of $k+1$-cores of rank r.

Thus for a fixed $\lambda \in \Pi^k$,

$$\begin{aligned}
h_r[X]\,\mathfrak{s}^{(k)}_\lambda[X] &= \sum_{\mu\in\mathcal{C}^{k+1}} |\overline{\mathcal{P}}^k(\mu,\lambda)| h_r[X]\, s^{(k)}_\mu[X] \\
&= \sum_{\mu\in\mathcal{C}^{k+1}} |\overline{\mathcal{P}}^k(\mu,\lambda)| \sum_{\text{rank } r \text{ weak strips } \rho/\mu} s^{(k)}_\rho[X] \\
&= \sum_{\rho\in\mathcal{C}^{k+1}} s^{(k)}_\rho[X] \sum_{\text{rank } r \text{ reverse maximal strips } \rho/\mu} |\overline{\mathcal{P}}^k(\mu,\lambda)| \\
&= \sum_{\rho\in\mathcal{C}^{k+1}} s^{(k)}_\rho[X] \sum_{\text{rank } r \text{ maximal strips } \nu/\lambda} |\overline{\mathcal{P}}^k(\rho,\nu)| \\
&= \sum_{\text{rank } r \text{ maximal strips } \nu/\lambda} \mathfrak{s}^{(k)}_\nu[X].
\end{aligned}$$

In the third equality we used Proposition 73, and in the fourth equality we used Theorem 75. $\square$

Remark 76. Suppose that strips of rank k are allowed. The results of this paper hold with a few minor changes.[1] For instance, Theorem 75 and Theorem 5 are still valid (with the case $r = k$ being allowed in Theorem 5). However, as the rest of the remark should make clear, the extension of Theorem 4 is somewhat more subtle.

When $\nu = \emptyset$ (and thus also $\mu = \emptyset$), the bijection on which Theorem 75 relies, associates to a reverse-maximal tableaux S a pair $(T, [\mathbf{q}])$, where T is a maximal tableau of a given shape ρ. If strips of rank k are allowed then Proposition 73 is not valid anymore, as adding a strip of rank k on a k-core does not produce a k-core. Therefore, if the weight of S has entries of size k, then the pushout of S does not produce a weak tableau T and Theorem 4 ceases to be valid. In the following, we will extend Theorem 4 to the case when strips of rank k are allowed.

The fact that $\mathrm{Weak}^{(k-1)}_\rho[X]$ is a symmetric function is not sufficient anymore to prove that $\mathfrak{S}^{(k)}_\mu[X]$ is a symmetric function. By Theorem 4, the sum of the terms that do not involve any power x_i^k in $\mathfrak{S}^{(k)}_\mu[X]$ is a symmetric function. Furthermore, we have that if $\mathrm{rs}(\mu)_1 < k$ then $\mathfrak{S}^{(k)}_\mu[X]$ does not involve any power x_i^k and thus $\mathfrak{S}^{(k)}_\mu[X]$ is a symmetric function in that case (see the proof of Proposition 79). Now if μ is a k-shape such that $\mathrm{rs}(\mu)_1 = k$, then by Lemma 78, μ has a unique reverse maximal strip of rank k. In this manner, it is not too difficult to see that the sum of the terms in $\mathfrak{S}^{(k)}_\mu[X]$ that involve powers of x_i^k is equal to $B_k\,\mathfrak{S}^{(k)}_\lambda[X]$, where $B_k m_\beta = m_{(k,\beta)}$ and is thus a symmetric function by induction. This proves that $\mathfrak{S}^{(k)}_\mu[X]$ is also a symmetric function if strips of rank k are allowed.

Finally, to complete the extension of Theorem 4, let $\pi^{(k)}$ be the projection onto Λ/I_{k-1}. Then for $\mu \in \Pi^k$, the cohomology k-shape function $\mathfrak{S}^{(k)}_\mu[X]$ has the

[1] However, the concept of lower augmentable corner which will be introduced in §§4.6 needs to be slightly modified: we define an augmentable corner b of a strip $S = \mu/\lambda$ as usual, except we disallow the case that b lies in a row of S that already contains k cells.

decomposition

$$\pi^{(k)}(\mathfrak{S}_\mu^{(k)}[X]) = \sum_{\rho\in\mathcal{C}^k} |\overline{\mathcal{P}}^k(\mu,\rho)|\, \mathrm{Weak}_\rho^{(k-1)}[X] \tag{52}$$

The result holds trivially from Theorem 4 since the projection will kill every $x^{\mathrm{wt}(T)}$ such that T has a strip of rank k.

4.4. **Elementary properties of $\mathfrak{S}_\lambda^{(k)}[X]$ and $\mathfrak{s}_\lambda^{(k)}[X]$.**

Proposition 77. *For $\lambda\in\Pi^k$, let λ^r (resp. λ^c) be the unique element of $\mathcal{C}^{k+1}$ such that $\mathrm{rs}(\lambda)=\mathrm{rs}(\lambda^r)$ (resp. $\mathrm{cs}(\lambda)=\mathrm{cs}(\lambda^c)$). Then one has*

$$\mathfrak{s}_\lambda^{(k)}[X] = s_{\lambda^r}^{(k)}[X] + \sum_{\rho\in\mathcal{C}^{k+1};\, \mathrm{rs}(\rho)\rhd\mathrm{rs}(\lambda)} |\overline{\mathcal{P}}^k(\rho,\lambda)|\, s_\rho^{(k)}[X] \tag{53}$$

and, similarly,

$$\mathfrak{s}_\lambda^{(k)}[X] = s_{\lambda^c}^{(k)}[X] + \sum_{\rho\in\mathcal{C}^{k+1};\, \mathrm{cs}(\rho)\rhd\mathrm{cs}(\lambda)} |\overline{\mathcal{P}}^k(\rho,\lambda)|\, s_\rho^{(k)}[X] \tag{54}$$

Proof. We will only prove (53), since (54) follows similarly. As already mentioned at the beginning of §§2.5, if m is a column move (resp. row move) from ν to μ, then $\mathrm{rs}(\nu)\rhd\mathrm{rs}(\mu)$ (resp. $\mathrm{rs}(\nu)=\mathrm{rs}(\mu)$) in the dominance order on partitions. Since λ is obtained from ρ by a sequence of moves, it only remains to show that $|\overline{\mathcal{P}}^k(\lambda^r,\lambda)|=1$. That is, that there exists a unique equivalence class of paths in the k-shape poset from λ^r to λ. Or equivalently, that there exists a unique equivalence class of paths in the k-shape poset from λ to λ^r. The proof is analogous to the proof that given μ/λ a strip, there exists a unique equivalence class of paths in Strip^μ to the reverse-maximal strip μ/ν (see Proposition 169). □

Let μ be a k-shape. The *surface strip* μ/λ of μ is the horizontal strip consisting of the topmost cell of each column of μ.

Lemma 78. *The surface strip of μ is the unique reverse maximal strip of μ with rank $\mathrm{rs}(\mu)_1$.*

Proof. It follows from the definitions and the fact that μ is a k-shape that the skew shape $\mathrm{Int}(\mu)/\mathrm{Int}(\lambda)$ is the surface strip of $\mathrm{Int}(\mu)$. Thus $\mathrm{rs}(\lambda)$ is obtained from $\mathrm{rs}(\mu)$ by removing the first row, and $\mathrm{cs}(\lambda)$ is obtained from $\mathrm{cs}(\mu)$ by reducing the last $\mathrm{rs}(\mu)_1$ columns each by 1. In particular, μ/λ is a strip. It is clear that the surface strip $S=\mu/\lambda$ is reverse maximal.

Let $S'=\mu/\nu$ be another reverse maximal strip with rank $\mathrm{rs}(\mu)_1$. The modified columns of S' must be exactly the last $\mathrm{rs}(\mu)_1$ columns, and furthermore, $\mathrm{rs}(\nu)=\mathrm{rs}(\lambda)$. It follows that $\nu=\lambda$. □

Proposition 79. *Let $\lambda\in\Pi^k$ be such that $\mathrm{rs}(\lambda)=\nu$. If we allow strips of rank k, then*

$$\mathfrak{S}_\lambda^{(k)}[X] = m_\nu + \sum_{\mu\lhd\nu} \tilde{K}_{\nu\mu} m_\mu \tag{55}$$

for some coefficients $\tilde{K}_{\nu\mu}\in\mathbb{Z}_{\geq 0}$.

Proof. Let $T \in \widetilde{\mathrm{Tab}}^k_\lambda$, and suppose that $\mathrm{wt}(T) = \mu$. Since $T = \emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)} = \lambda$ is a sequence of strips, we have in particular that $\mathrm{rs}(\lambda^{(i)})/\mathrm{rs}(\lambda^{(i-1)})$ is a horizontal μ_i-strip for all i. This gives immediately that $\mu \trianglelefteq \nu$ (think of the triangular expansion of the homogeneous symmetric functions into Schur functions). Now, the unique $T \in \widetilde{\mathrm{Tab}}^k_\lambda$ such that $\mathrm{wt}(T) = \nu$ is obtained by recursively taking the surface strips of λ. Finally, $\tilde{K}_{\nu\mu} \in \mathbb{Z}_{\geq 0}$ by definition of $\mathfrak{S}^{(k)}_\lambda[X]$. ☐

Proposition 80. *Let $\lambda \in \Pi^k$, and let $\omega : \Lambda \to \Lambda$ be the homomorphism that sends the r^{th} complete symmetric function to the r^{th} elementary symmetric function. Then*

$$\omega(\mathfrak{s}^{(k)}_\lambda[X]) = \mathfrak{s}^{(k)}_{\lambda'}[X] \tag{56}$$

and

$$\omega(i_{k-1}(\mathfrak{S}^{(k)}_\lambda[X])) = \mathfrak{S}^{(k)}_{\lambda'}[X] \mod I_{k-1} \tag{57}$$

where λ' is the conjugate of λ, and where i_k was defined in (11).

Proof. For the proof of (56) we proceed by induction. The result holds for k large since in that case $s^{(k)}_\lambda[X] = s_\lambda[X]$ is a usual Schur functions and it is known that $\omega(s_\lambda[X]) = s_{\lambda'}[X]$. From (18) when $t = 1$ we get

$$\omega(\mathfrak{s}^{(k)}_\lambda[X]) = \sum_{\rho \in \mathcal{C}^{k+1}} |\overline{\mathcal{P}}^k(\rho,\lambda)|\, \omega(s^{(k)}_\rho[X]) \tag{58}$$

Since $s^{(k)}_\rho[X] = \mathfrak{s}^{(k+1)}_\rho[X]$ from (19), we can suppose by induction that $\omega(s^{(k)}_\rho[X]) = s^{(k)}_{\rho'}[X]$. We also have $|\overline{\mathcal{P}}^k(\rho,\lambda)| = |\overline{\mathcal{P}}^k(\rho',\lambda')|$ by the transposition symmetry of the k-shape poset, and thus (56) follows from (58).

For the proof of (57), we have from Theorem 4 that

$$\omega(i_{k-1}(\mathfrak{S}^{(k)}_\mu[X])) = \sum_{\rho \in \mathcal{C}^k} |\overline{\mathcal{P}}^k(\mu,\rho)|\, \omega(i_{k-1}(\mathrm{Weak}^{(k-1)}_\rho[X])) \tag{59}$$

The duality (11) between k-Schur functions and dual k-Schur functions implies that

$$\omega(i_{k-1}(\mathrm{Weak}^{(k-1)}_\rho[X])) = \mathrm{Weak}^{(k-1)}_{\rho'}[X] \mod I_{k-1} \tag{60}$$

given that $\omega(s^{(k-1)}_\rho[X]) = s^{(k-1)}_{\rho'}[X]$ (see [13]) and that ω is an isometry. The result then follows from (59) since, as we saw earlier, $|\overline{\mathcal{P}}^k(\mu',\rho')| = |\overline{\mathcal{P}}^k(\mu,\rho)|$. ☐

4.5. **Basics on strips.** The remainder of this section deals with the properties of strips and augmentation moves. Sections 5 and 6 study pushouts involving row and column moves respectively.

The next results help in checking whether something is a strip.

Property 81. *Let $S = \mu/\lambda$ be a horizontal strip of k-shapes and c a column which contains a cell of S. Then $\mathrm{cs}(\mu)_c \geq \mathrm{cs}(\lambda)_c$.*

Proof. Let $b \in \partial\lambda$ be in column c and b' be the cell just above b. Since μ/λ is a horizontal strip, $k \geq h_\lambda(b) \geq h_\mu(b')$. This implies $b' \in \partial\mu$ and the result follows since there is a cell of S in column c. ☐

Property 82. *Suppose $S = \mu/\lambda$ is a strip. Then $\partial\lambda \setminus \partial\mu$ is a horizontal strip.*

Proof. The lemma follows from Property 81 and the fact that μ/λ is a horizontal strip. ☐

Lemma 83. *Let μ/λ be a strip and c be such that $cs(\mu)_c = cs(\lambda)_c + 1 \le \mathrm{cs}(\lambda)_{c-1}$. Then there is a cover-type λ-addable string s such that $c_{s,d} = c$, $\lambda \cup s \in \Pi^k$, and $\mu/(\lambda \cup s)$ is a strip.*

Proof. Let b be the unique cell in column c of μ/λ. The hypotheses imply that b is λ-addable. Let s be the maximal λ-addable string such that $s \subset \mu$ and s ends with b. Say the top cell y of s is in row i. Let $x = (i,j) = \mathrm{left}_i(\partial\lambda)$. Let b' be the λ-addable cell in column j, if it exists.

The string s is of row-type or cover-type by the hypotheses. Suppose s is of row-type. Then $h_\lambda(x) = k$. Then $b' \cup s$ is a λ-addable string. Since μ/λ is a strip we have $\mathrm{cs}(\mu)_j \ge \mathrm{cs}(\lambda)_j$. But $x = \mathrm{bot}_j(\partial\lambda) \notin \partial\mu$. Hence $b' \in \mu$, contradicting the maximality of s.

Therefore s is of cover-type. Since μ/λ is a strip, $\mathrm{rs}(\mu)_i \ge \mathrm{rs}(\lambda)_i$. Suppose $\mathrm{rs}(\mu)_i = \mathrm{rs}(\lambda)_i$, and let μ/λ have ℓ cells in row i. By supposition, there are also ℓ cells of $\partial\lambda \setminus \partial\mu$ in row i. Since $\mathrm{cs}(\lambda) \subseteq \mathrm{cs}(\mu)$ and μ/λ is a horizontal strip, there must then be cells of μ/λ in columns $j, \dots, j+\ell-1$ that are contiguous to the ℓ cells of μ/λ in row i. In particular, the λ-addable corner b' is contiguous to y. Again $b' \cup s$ is a λ-addable string, contradicting the maximality of s.

Therefore $\mathrm{rs}(\mu)_i > \mathrm{rs}(\lambda)_i$, which ensures that $\mathrm{rs}(\mu)_i \ge \mathrm{rs}(\lambda \cup s)_i$. Since $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip we deduce that $\lambda \cup s$ is a k-shape. It then easily follows that $\mu/(\lambda \cup s)$ is a strip. □

Corollary 84. *Any strip $S = \mu/\lambda$ of rank ρ has a decomposition into ρ cover-type strings. More precisely, for every sequence $c_1, c_2, \dots, c_\rho$ of modified columns of S such that $c_i < c_j$ if $\mathrm{cs}(\lambda)_{c_i} = \mathrm{cs}(\lambda)_{c_j}$, there is a chain in Π: $\lambda = \lambda^{(0)} \subset \cdots \subset \lambda^{(\rho)} = \mu$ such that $t_i = \lambda^{(i)}/\lambda^{(i-1)}$ is a $\lambda^{(i-1)}$-addable cover-type string with modified column c_i.*

Proof. Follows by induction from Lemma 83. □

Corollary 85. *Let $S = \mu/\lambda$ be a strip. If a column contains a cell in $\partial\lambda \setminus \partial\mu$ then it also contains a cell of S.*

Proof. Each such cell is a removed cell for one of the cover-type strings that constitute S. □

Lemma 86. *Suppose $S = \mu/\lambda$ is a strip, and let $s = \{a_1, \dots, a_\ell\} \subseteq S$ be a λ-addable string (with a_1 the topmost). For each $i \in [1,\ell]$ let r_i (resp. c_i) be the row (resp. column) of a_i. Then*

(1) *If $i < j$ then there are at least as many cells of S in row r_j than there are in row r_i.*
(2) *If $i > j$ then there are at least as many μ-addable cells in column c_j than there are in column c_i.*

Proof. For (1), let r_i and r_{i+1} violate the first assertion, and let b be the the rightmost cell of S in row r_i. It is then easy to see that in μ we have the contradiction that the column of b is larger than the column of the first cell of S in row r_i.

For (2), it is enough to consider the case c_i and c_{i+1}. Since a_i and a_{i+1} are contiguous, $\mathrm{left}(\partial\lambda)_{r_{i+1}}$ lies in column c_i. Suppose that column c_{i+1} has $p \ge 1$ μ-addable cells, and let $b = \mathrm{bot}(\partial\mu)_{c_i^-}$ lie in row R. Then row R is at least p rows above row r_{i+1} since otherwise $\mathrm{rs}(\mu)_R$ would be larger than $\mathrm{rs}(\lambda)_{r_{i+1}}$ contradicting

the fact that $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. Since $\mathrm{cs}(\mu)_{c_i^-} \geq \mathrm{cs}(\mu)_{c_i}$, column c_i needs to have at least p μ-addable cells. $\square$

4.6. **Augmentation of strips.** We first observe the following:

Remark 87. The negatively modified columns (resp. rows) of an augmentation move of the strip S are positively modified columns (resp. rows) of S.

Property 88. *All augmentation column moves of a strip $S = \mu/\lambda$ have rank 1.*

Proof. If it were not the case, the modified rows of m (which all have the same length by definition) would violate the condition that $\mathrm{rs}(m * \mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. $\square$

Let $S = \mu/\lambda$ be a strip. A μ-addable cell a is called

(1) a *lower augmentable corner* of S if adding a to μ removes a cell from $\partial\mu$ in a modified column c of S in the same row as a.
(2) an *upper augmentable corner* of S if a does not lie on top of any cell in S and adding a to μ removes a cell from $\partial\mu$ in a modified row r of S and in the same column as a.

We say that a is *associated* to c (or r, respectively). We call a modified column c of S *leading* if the cell $c \cap S$ (the cell of S in column c) is leftmost in its row in S.

Lemma 89. *Let $S = \mu/\lambda$ be a strip. Then any augmentation move m contains an augmentable corner of S.*

Proof. If the strip S admits an augmentation row (resp. column) move m then the top left (resp. bottom right) cell of m is a lower (resp. upper) augmentable corner of S. $\square$

Definition 90. A *completion row move* is one in which all strings start in the same row. It is maximal if the first string cannot be extended below. A *quasi-completion column move* is a column augmentation move from a strip S that contains no lower augmentable corner. A *completion column move* is a quasi-completion move from a strip S that contains no upper augmentable corner below its unique (by Property 88) string [2]. A completion column move or a quasi-completion column move is maximal if its string cannot be extended above. A completion move is a completion row/column move.

The definition of completion move is transpose-asymmetric since strips are. Our main result for augmentations of strips is the following. Its proof occupies the remainder of the section.

Proposition 91. *Let $S = \mu/\lambda$ be a strip.*

(1) *S has a unique maximal augmentation $S' \in \mathrm{Strip}_\lambda$.*
(2) *There is one equivalence class of paths in Strip_λ from S to S'.*
(3) *The unique equivalence class of paths in Strip_λ from S to S' has a representative consisting entirely of maximal completion moves.*

[2]The reason for distinguishing between completion and quasi-completion column moves will only become apparent in §7.2 (Lemma 151).

Let $m = s_1 \cup s_2 \cup \cdots \cup s_r$ be an augmentation row move from S. Then $c_{s_i,u}$ is a modified column of S for each $i \in [1, r]$. Since m is a move and $m * \mu \in \Pi$, the columns $\{c_{s_i,u} \mid i \in [1, r]\}$ are part of a group of modified columns of S and must be the rightmost r columns in this group, by Property 21.

Lemma 92. *Let s be a λ-addable row-type (resp. column-type) string that cannot be extended below (resp. above). Then $\mathrm{cs}(\lambda)_{c_{s,d}} < \mathrm{cs}(\lambda)_{c^-_{s,d}}$ (resp. $\mathrm{rs}(\lambda)_{r_{s,u}} < \mathrm{rs}(\lambda)_{r^-_{s,u}}$).*

Proof. Let $s = \{a_1, a_2, \ldots, a_\ell\}$ be a row-type string and suppose $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c^-}$ where $c = c_{s,d}$. We have $\lambda_{c^-} > \lambda_c$ since a_ℓ is λ-addable, and thus the cell immediately to the left of $b = bot_c(\partial\lambda)$ is not in $\partial\lambda$. This implies that $h_\lambda(b) \geq k-1$ so that $h_{\lambda\cup s}(b) = k$ given that s is a row-type string. By Remark 13, there is a λ-addable corner at the end of the row of b that is contiguous with a_ℓ, so that s can be extended below, a contradiction. The column-type case is similar. $\square$

Lemma 93. *Let $m = s_1 \cup s_2 \cup \cdots \cup s_r$ be a non-maximal completion row move from $\lambda \in \Pi$ and let $t_1 = s_1 \cup \{a_{\ell+1}, a_{\ell+2}, \ldots, a_{\ell+\ell'}\}$ be the maximal row-type string which extends s_1 below. Then there is a unique completion row move $n = t_1 \cup t_2 \cup \cdots \cup t_r$ from λ.*

Proof. By Proposition 27, if n exists, it is determined by t_1. We show that there are strings $t_2, t_3, \ldots, t_r$ that can be added to $\lambda \cup t_1$. Let $s_i = \{a_1^{(i)}, \ldots, a_\ell^{(i)}\}$ and $R = \mathrm{row}(a_{\ell+1})$. Since the cells $a_\ell^{(1)}, \ldots, a_\ell^{(r)}$ lie in the same row, we have by Lemma 31 that there is room for $a_i^{(1)}, \ldots, a_i^{(r)}$ in the row of $a_i = a_i^{(1)}$ for all $i = \ell+1, \ldots, \ell+\ell'$. These cells obviously all lie in columns of $\partial\lambda$ of the same length. Thus the result holds since Lemma 92 allows us to conclude that $n * \lambda \in \Pi$. $\square$

Lemma 94. *Suppose a is a lower augmentable corner in row R of the strip $S = \mu/\lambda$, associated to the column c.*

1. *$\mathrm{bot}_c(\partial\mu) = \mathrm{bot}_c(\partial\lambda) = (R, c)$ and $h_\mu(R, c) = k$.*
2. *c is a leading column.*
3. *Suppose that $c' > c$ is a leading column such that $\mathrm{cs}(\mu)_{c'} = \mathrm{cs}(\mu)_c$. Then there is a lower augmentable corner a' which is associated to c'.*
4. *Let r be the number of cells of S in the row of $\mathrm{top}_c(\mu)$. Then $\lambda_{R^-} \geq \lambda_R + r$.*

Proof. (1) and (2) are straightforward using the fact that c is a modified column of S. For (3), let $b = \mathrm{bot}_{c'}(\partial\mu)$. Then $h_\mu(b) \geq h_\mu(\mathrm{bot}_c(\partial\mu)) = k$ by (1). Thus the addable corner at the end of the row of b (assured to exist by Remark 13) must be lower augmentable. (4) is implied by Remark 13. $\square$

Lemma 95. *Suppose m is a non-maximal completion row move from a strip $S = \mu/\lambda$. Let $t_1 = s_1 \cup \{a_{\ell+1}, a_{\ell+2}, \ldots, a_{\ell+\ell'}\}$ be the maximal row-type string which extends the first string $s_1 = \{a_1, \ldots, a_\ell\}$ of m below. Then the completion row move n from μ of Lemma 93 is a maximal completion row move from S.*

Proof. We use the notation of Lemma 93. We first show that $n * S$ is a horizontal strip. Since $a_{\ell+1} = a_{\ell+1}^{(1)}$ is a lower augmentable corner of the strip $m * S$, by Lemma 94(4), the cells $\{a_{\ell+1}^{(i)} \mid i \in [1, r]\}$ lie above cells of λ. Now suppose that the cells $\{a_{\ell+j}^{(i)} \mid i \in [1, r]\}$ do not lie on any cell of S. Let R be the row of $\{a_{\ell+j+1}^{(i)} \mid i \in [1, r]\}$, and let c the column of $a_{\ell+j}^{(r)}$. Since $\mathrm{bot}_c(\partial\mu)$ lies in row R and

since there are no cells of S in column c by supposition, we have that $\mathrm{left}_{R-1}(\partial\lambda)$ is strictly to the right of column c by Corollary 85. Therefore, there are at least r extra cells of $\partial\mu$ in row R to the left of $\mathrm{left}_{R-1}(\partial\lambda)$. This implies that $\lambda_{R-1}-\mu_R \geq r$ since $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. This proves that $\{a^{(i)}_{\ell+j+1} \mid i \in [1,r]\}$ also do not lie on on any cell of S and we get by induction that S is a horizontal strip. Since n is a row move, $\mathrm{rs}(n*\mu)/\mathrm{rs}(\lambda) = \mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. Finally, since n removes cells in the same columns of $\partial\mu$ as m does and $n*S$ is a horizontal strip, $\mathrm{cs}(n*\mu)/\mathrm{cs}(\lambda)$ is a vertical strip. Hence $n*S$ is a strip in Π. □

Lemma 96. *Let $\lambda \in \mathcal{C}^k$ and $S = \mu/\lambda$ a strip with no lower augmentable corners. Suppose a is a μ-addable corner such that adding a to the shape μ removes a cell from $\partial\mu$ in a modified row r of S. Then a is an upper augmentable corner.*

Proof. We must show that a does not lie on top of any cell in S. Suppose otherwise. Let $b = \mathrm{bot}_{\mathrm{col}(a)}(\partial\mu)$. We have that $\mathrm{col}(a)$ is not a modified column of S, for otherwise $\mathrm{row}(b)$ has a lower augmentable corner for S, a contradiction.

Let b' be the cell immediately below b. Since $\mathrm{col}(a)$ is not a modified column but it contains a cell in S, we must have $b' \in \partial\lambda - \partial\mu$. Furthermore, Property 82 implies that $b' = \mathrm{left}(\partial\lambda)_{\mathrm{row}(b')}$. Since $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip we have $\mathrm{rs}(\lambda)_{\mathrm{row}(b')} \geq \mathrm{rs}(\mu)_{\mathrm{row}(b)}$ and hence that $h_\lambda(b') \geq h_\mu(b) = k$. But $b' \in \partial\lambda$ implies $h_\lambda(b') = k$, contradicting that $\lambda \in \mathcal{C}^k$. □

Example 97. The k-core condition in Lemma 96 is necessary. For $k = 4$ consider

$$\begin{array}{cccc} a & & & \\ \hline \multicolumn{1}{|c|}{S} & & & \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{S} & \\ \hline \multicolumn{1}{|c|}{b'} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{S} \\ \hline \end{array}.$$

Lemma 98. *Let a be a lower augmentable corner of a strip $S = \mu/\lambda$ associated to column c. Let S contain r cells in the row containing the cell $c \cap S$. Suppose that a is chosen rightmost amongst augmentable corners associated to columns of the same size in $\partial\mu$. Let $t_1 = \{a = a_1, a_2, \ldots, a_{\ell'}\}$ be the maximal row type string which extends a below. Then there is a maximal completion row move n from S which has rank r and initial string t_1.*

Proof. We apply the construction in Lemma 95 with m an empty move. m is not maximal since there is a lower augmentable corner a in some row R, which can be extended to a row-type string by Lemma 30. The move m has rank r since r cells can be added to row R of λ by Lemma 94(4). The choice of a guarantees that the negatively modified columns of n have the same size and that the monotonicity of column sizes is preserved. The argument in Lemma 95 completes the proof. □

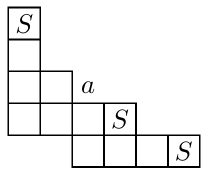
Lemma 99. *Let $S = \mu/\lambda$ be a strip with $t > 1$ lower augmentable corners and m an augmentation row move from S. Then there is a maximal completion row move M from S such that (m, M) admits an elementary equivalence $\tilde{m}M \equiv \tilde{M}m$ in Strip_λ, $\tilde{m}$ contains $t-1$ lower augmentable corners, and $\tilde{M}$ is a maximal completion row move.*

Proof. Let a be the rightmost lower augmentable corner of S inside m (it exists by Lemma 89). Then define M to be the move from S that arises from a as described

in Lemma 98. The strings of m and M containing a both start at a. By Lemma 38, m and M are matched above.

If m and M are matched below, it follows from the proof of Proposition 51 that $m = M$. This is a contradiction since M contains only one augmentable corner of S. Therefore M continues below m and the pair (m, M) is a Case (3) of an elementary row equivalence: $\tilde{m}$ contains all the lower augmentable corners of m apart from a; $\tilde{M}$ contains a lower part of M. The claimed properties follow immediately. □

Example 100. Column completions behave somewhat differently: it is not always possible to choose the maximal extension of an upper augmentable corner, e. g.,



with $k = 5$.

Lemma 101. *Let $S = \mu/\lambda$ be a strip and let $c = \mathrm{left}_{R^-}(\partial\lambda)$ and $b = \mathrm{left}_R(\partial\mu)$ for some row R. Suppose that $\mathrm{rs}(\mu)_R = \mathrm{rs}(\lambda)_{R^-}$. Then c and b lie in the same column if one of the following conditions is satisfied.*

(1) $h_\mu(b) < k - 1$.
(2) *There is a cell of S in the column of b and $h_\mu(b) = k - 1$.*
(3) *There is no upper augmentable corner of S in the column of b, there is no lower augmentable corner of S in row R, row R is modified by S and $h_\mu(b) = k$.*

Proof. c cannot be to the left of b by Property 82. Suppose that c is to the right of b. Let c' be the cell left-adjacent to c.

Case (1). Since $h_\mu(b) < k - 1$ and $\mathrm{rs}(\mu)_R = \mathrm{rs}(\lambda)_{R^-}$, we have the contradiction that $h_\lambda(c') \leq 2 + h_\mu(b) \leq k$.

Case (2). Since there is a cell of S in the column of b, any column of λ to the right of b is shorter than the column of c in μ. Given that $h_\mu(b) = k - 1$ and $\mathrm{rs}(\mu)_R = \mathrm{rs}(\lambda)_{R^-}$, we have the contradiction that $h_\lambda(c') \leq 1 + h_\mu(b) = k$.

Case (3). By hypothesis $h_\mu(b) = k$. If there is a cell of S in $\mathrm{col}(b)$ then $\mathrm{col}(b)$ is a modified column of S and we have the contradiction that there is a lower augmentable corner of S in row R ($\lambda_{R^-} > \mu_R$ by hypothesis). Otherwise we get the contradiction that there is an upper augmentable corner of S associated to row R in $\mathrm{col}(b)$. □

Lemma 102. *Let $S = \mu/\lambda$ be a strip without lower augmentable corners and let a be an upper augmentable corner of S. Let $s = \{a_1, \dots, a_\ell = a\}$ be the maximal extension of a above, subject to the condition that the a_i do not lie on top of cells of S. Then $m = s$ is a quasi-completion column move from S.*

Proof. Let $\mathrm{row}(a_1) = R$, $b = (R, c) = \mathrm{left}_R(\partial\mu)$ and $d = \mathrm{left}_{R^-}(\partial\lambda)$. Suppose first that s is the maximal extension of a without being constrained by not lying on top of S. By Lemmata 30 and 92, s is a column type string and $m * \mu$ is a k-shape. It suffices to show that $\mathrm{rs}(m * \mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. Since S is a strip, $\mathrm{rs}(\lambda)_{R^-} \leq \mathrm{rs}(\mu)_{R^-} = \mathrm{rs}(m*\mu)_{R^-}$, so it remains to show that $\mathrm{rs}(m*\mu)_R \leq \mathrm{rs}(\lambda)_{R^-}$. The only way this would fail is if $\mathrm{rs}(\mu)_R = \mathrm{rs}(\lambda)_{R^-}$. Since s is maximal, we have

$h_b(\mu) < k-1$. Thus from Lemma 101(1), we have that b and d lie in the same column. Since $\mathrm{rs}(\mu)_R = \mathrm{rs}(\lambda)_{R^-}$ this gives the contradiction that a_1 lies over a cell of S.

Now suppose that s is blocked from extending further by the constraint of not lying on top of S. Consider first the case that $h_\mu(b) = k-1$, and observe that, as in the previous case, if $m * \mu$ fails to be a k-shape or $\mathrm{rs}(m*\mu)/\mathrm{rs}(\lambda)$ fails to be a horizontal strip, then $\mathrm{rs}(\mu)_R = \mathrm{rs}(\lambda)_{R^-}$ (S is a strip and thus $\mathrm{rs}(\mu)_{R-} \geq \mathrm{rs}(\lambda)_{R-} \geq \mathrm{rs}(\mu)_R$). Lemma 101(2) then implies that b and d lie in the same column and the result follows from the argument given in the previous case. Finally, consider the case that $h_\mu(b) = k$. Column c is not a modified column of S since a_1 cannot be a lower augmentable corner. Thus the cell b' below b is in $\partial\lambda$ and so is the cell below a_1. This gives the contradiction $h_{b'}(\lambda) > h_b(\mu) = k$. □

4.7. **Maximal strips for cores.** Recall that a strip S is maximal if it does not admit any augmentation move.

Proposition 103. *A strip is maximal if and only if it has no augmentable corners.*

Proof. By Lemma 89, if the strip S admits an augmentation move then S has an augmentable corner. Conversely, if S has an augmentable corner, then S admits a maximal completion move by Lemmata 98 and 102. □

Lemma 104. *Let $S = \mu/\lambda$ be a maximal strip and let c, c' be two modified columns such that $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c'}$. Then the cells $S \cap c$ and $S \cap c'$ are on the same row.*

Proof. Suppose otherwise. We may assume that $c' = c+1$. Let $b' = bot_{c'}(\partial\lambda)$ and b be the cell just below $bot_c(\partial\lambda)$. Then

$$h_\mu(b') \geq h_\lambda(b') + 1 \geq h_\lambda(b) - 1 \geq k\,.$$

Since c' is a modified column, $b' \in \partial\mu$, that is, $h_\mu(b') = k$. But then there must be a lower augmentable corner for S at the end of the row of b', contradicting Proposition 103.

□

Proposition 105. *Suppose $S = \mu/\lambda$ is a maximal cover and $\lambda \in \mathcal{C}^k$. Then $\mu \in \mathcal{C}^k$.*

Proof. It suffices to check $h_\mu(x)$ for cells x in the modified row or column, such that $h_\mu(x) = h_\lambda(x) + 1$. For the modified row r, let $b = \mathrm{left}_r(\partial\lambda)$. Then $h_\lambda(b) < k-1$, for otherwise S is not maximal. All cells to the left of b have $h_\lambda > k$. Similar reasoning applies to the modified column. □

Proposition 106. *Suppose $S = \mu/\lambda$ is a maximal strip and $\lambda \in \mathcal{C}^k$. Then $\mu \in \mathcal{C}^k$.*

Proof. By Proposition 105 it suffices to show that S can be expressed as a sequence of maximal covers. Construct a sequence of covers for S using Lemma 83. By Proposition 103, S has no augmentable corner. We claim that this implies that the successive covers constructed have no augmentable corners which would then imply their maximality. Note that a modified row or column of one of these covers is immediately also one of S.

Let $C = \nu/\kappa$ be such a cover. For lower augmentable corners, this is clear since such augmentable corners are augmentable corners of S. For an upper augmentable corner $a \notin S$ of C, we apply Lemma 96 which implies that a is an upper augmentable corner of S. □

4.8. **Equivalence of maximal augmentation paths.** Let $S = \mu/\lambda$ be a strip. Suppose m and M are distinct augmentation moves from S. We say that the pair (m, M) defines an augmentation equivalence if there is an elementary equivalence of the form $\tilde{M}m \equiv \tilde{m}M$ such that $\tilde{m}$ and $\tilde{M}$ are augmentation moves from the strips $M * S$ and $m * S$ respectively. Note that given the elementary equivalence, $\tilde{m}$ and $\tilde{M}$ are augmentation moves if and only if $\tilde{M} * m * S$ (or $\tilde{m} * M * S$) is a strip.

Lemma 107. *Suppose m and M are respectively a maximal completion row move and an augmentation column move from a strip S. Then (m, M) defines an augmentation equivalence. Moreover,*

(1) *If $m \cap M = \emptyset$ then no cell of m is contiguous to a cell of M.*
(2) *If $m \cap M \neq \emptyset$ then m continues above and below M.*

Proof. Let $S = \mu/\lambda$. For (1) the non-contiguity follows from the maximality of m. The other assertions follow easily in this case.

So let $m \cap M \neq \emptyset$. By Property 88, M consists of a single μ-addable column-type string t. By Property 26 and Lemma 35 the first string s of m must be the unique string that meets M. We claim that m continues above and below M. Consider the highest cell $x \in m \cap M$. Suppose x is the highest cell in s. Let $b = \mathrm{left}_{\mathrm{row}(x)}(\partial\mu)$. By Definition 17 $h_\mu(b) = k$. x is a lower augmentable corner of S so that b lies in a modified column of S. By Property 56, M and m cannot be matched above and thus we get the contradiction that M needs to continue above m with an element in the column of b lying on top of S. Therefore m continues above M. Now consider the lowest cell $y \in m \cap M$. Suppose y is the lowest cell in s. By Definition 17, $h_\mu(\mathrm{bot}_{\mathrm{col}(y)}(\partial\mu)) \leq k-1$. Again by Property 56, M and m cannot be matched below and thus y is not the lowest cell of t, which gives $h_\mu(\mathrm{bot}_{\mathrm{col}(y)}(\partial\mu)) = k-1$. But then s can be extended below, contradicting the maximality of m. Therefore m continues above and below M. It is straightforward to check that in this case, the resulting elementary equivalence $\tilde{M}m \equiv \tilde{m}M$, when applied to S, ends at a strip. □

Lemma 108. *Suppose m and M are distinct maximal completion row moves from the strip $S = \mu/\lambda$. Then (m, M) defines an augmentation equivalence. Moreover, $m \cap M = \emptyset$ and exactly one of the following holds:*

(1) *m and M do not interfere.*
(2) *(m, M) is interfering and lower-perfectible with added cells m_{per} such that $m \cup m_{\mathrm{per}}$ is a maximal completion row move from the strip $M * S$ and $M \cup m_{\mathrm{per}}$ is a maximal completion row move from $m * S$.*
(3) *The same as (2) with the roles of m and M interchanged.*

Proof. Suppose that $m \cap M \neq \emptyset$ and $m \neq M$. Then by maximality we may without loss of generality assume that m continues above M but (m, M) is matched below. But m must contain a cell in a modified column associated to M, contradicting the assumption that $S \cup m$ is a strip.

Therefore $m \cap M = \emptyset$. We may assume that m and M interfere, and that m is above M. Let c be the column such that $\mathrm{cs}(\mu \cup m \cup M)_c = \mathrm{cs}(\mu \cup m \cup M)_{c^-} + 1$. Then m adds the cell atop column c of $\partial\mu$ and M removes the cell $(r, c^-) = \mathrm{bot}_{c^-}(\partial\mu)$. Let $x = \mathrm{left}_r(\partial\mu)$ and $y = \mathrm{bot}_c(\partial\mu) = (r', c)$. By Definition 17 $h_\mu(x) = k$ and $h_\mu(y) \leq k-1$.

Suppose $r > r'$. We have $\mathrm{rs}(\mu)_{r'} \geq \mathrm{rs}(\mu)_r$ and $\mathrm{cs}(\mu)_{\mathrm{col}(x)} = \mathrm{cs}(\mu)_{c^-} = \mathrm{cs}(\mu)_c + 1$, so that $h_\mu(y) \geq h_\mu(x) - 1 = k - 1$. Therefore $h_\mu(y) = k - 1$. By Remark 13 there is a μ-addable cell in row r, which is below and contiguous with the cell of m in column c. This contradicts the maximality of m. Therefore $r = r'$ and $y = (r, c)$.

Let $\nu = \mu \cup M$. We have $h_\nu(y) = k - 1$. Since the negatively modified columns of M and the positively modified columns of m have their lowest k-bounded cell in the same row and $\mathrm{rs}(\nu)_{r^-} \geq \mathrm{rs}(\nu)_r$, we deduce that $\nu_{r^-} - \nu_r \geq \mathrm{rank}(m) + \mathrm{rank}(M)$. Using this and the maximality of M, by Lemma 93 we may deduce that viewing m as ν-addable, each of its strings can be maximally extended below to contain a cell in each of the rows of M by Lemma 31. Call the added cells m_{per}. It is straightforward to verify the remaining assertions. □

Lemma 109. *Suppose m and M are distinct maximal quasi-completion column moves for the strip S. Then (m, M) defines an augmentation equivalence. Moreover, exactly one of the following holds:*

1. *$m \cap M = \emptyset$ and m and M do not interfere.*
2. *$m \cap M \neq \emptyset$ and either $m \subset M$ or $M \subset m$.*

Proof. Suppose that $m \cap M = \emptyset$ and there is interference. Recall that m and M are of rank 1 and without loss of generality we can suppose that M is above m. Then the highest cell of m is in a row R such that R^- is a positively modified row of S by Remark 87 (since R^- is a negatively modified row of M), and such that $\mathrm{rs}(m * \mu)_R = \mathrm{rs}(m * \mu)_{R^-}$. We thus have the contradiction that $\mathrm{rs}(m * \mu)/\mathrm{rs}(\lambda)$ is not a horizontal strip.

If $m \cap M \neq \emptyset$, then by maximality they finish at the same point above. Given that both are of rank 1, we deduce that $m \subset M$ or $M \subset m$. □

We now prove Proposition 91.

Proof. Let S be a strip such that

$$\text{the result holds for any proper augmentation of } S. \tag{61}$$

Let $(m_1, m_2, \ldots, m_x)$ and $(M_1, M_2, \ldots, M_y)$ be distinct augmentation paths from S to maximal strips. If $m_1 = M_1$ we are done by induction. So suppose $m_1 \neq M_1$. If m_1 and M_1 are maximal completion moves then by Lemmata 107, 108 and 109, the pair (m_1, M_1) defines an augmentation equivalence $\tilde{M}_1 m_1 \equiv \tilde{m}_1 M_1$. By (61) there are equivalences of augmentation paths of the form

$$m_x \cdots m_2 m_1 \equiv \cdots \tilde{M}_1 m_1 \equiv \cdots \tilde{m}_1 M_1 \equiv M_y \cdots M_2 M_1.$$

It thus suffices to show that any augmentation path $(m_1, m_2, \ldots, m_x)$ ending at a maximal strip, is equivalent to one which begins with a maximal completion move. If m_1 is a non-maximal completion row move, then Lemma 95 implies that $m_1 \subset m$ where m is a maximal completion row move with the same lower augmentable corner. But then $(m \backslash m_1)(m_1) \equiv m$ is a row equivalence and using (61) we deduce that $(m_1, m_2, \ldots, m_x)$ is equivalent to a path beginning with m. A similar argument works for the column case.

We may thus assume that m_1 is a non-completion augmentation row or column move. In the case of the non-completion augmentation row move, the argument is completed by Lemma 99. In the case of the non-completion augmentation column move, S either contains some lower augmentable corners or some upper augmentable corners above the one associated to m_1. In the former case, let M be the maximal

completion row move associated to a lower augmentable corner a of S such as described in Lemma 98. By Lemma 107, the argument is completed in that case. In the latter case, let M be the maximal completion column move associated to the highest upper augmentable corner a of S such as described in Lemma 102. The lemma then follows from Lemma 109.

□

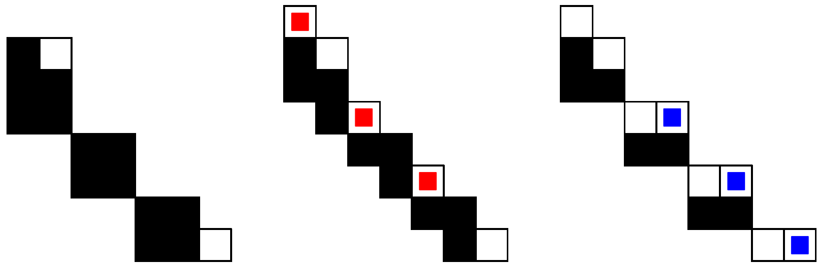
4.9. **Canonical maximization of a strip.** Let $S = \mu/\lambda$ be a strip. The following algorithm `MaximizeStrip` produces an augmentation path $\mathbf{q} = (\mu = \mu^0 \to \mu^1 \to \cdots \to \mu^M = \rho)$ in Strip_λ ending at a maximal strip ρ/λ. This path is comprised of maximal completion moves; the existence of such a path is asserted by Proposition 91(3).

proc `MaximizeStrip`(μ, λ):
    local $\rho := \mu, q := (\mu)$
    while True:
        if the strip ρ/λ has a lower augmentable corner:
            let x be the rightmost one
            let s be the maximal ρ-addable string extending x below
            $\rho := \rho \cup s$
            append ρ to q
            continue
        if the strip ρ/λ has an upper augmentable corner:
            let x be the rightmost one
            let s be the maximal ρ-addable string extending x above,
                subject to not having a cell atop ρ/λ
            $\rho := \rho \cup s$
            append ρ to q
            continue
        break
    return q

The path $\mathbf{q}$ is initialized to be the path of length zero starting and ending at μ and the current strip ρ/λ is initialized to be μ/λ. Whenever the current strip ρ/λ has a lower augmentable corner, the algorithm appends a completion row move to $\mathbf{q}$ by Lemma 98 and applies the move to ρ. Whenever the current strip ρ/λ has no lower augmentable corner but an upper augmentable one, the algorithm appends a completion column move m to $\mathbf{q}$ by Lemma 102 and applies the move to ρ. When ρ/λ has no augmentable corners, by Proposition 103 the algorithm terminates with ρ/λ a maximal strip and returns the current path $\mathbf{q}$.

Example 110. Let $k = 4$, $\lambda = (6, 6, 4, 4, 2, 2, 1)$, and $\mu = (7, 6, 4, 4, 2, 2, 2)$. Calling `MaximizeStrip` with the strip μ/λ, the output path $\mathbf{q} = (\mu = \mu^0, \mu^1, \mu^2 = \rho)$ is

given below. The boxes of $\partial\lambda \cap \partial\mu^i$ are black and the rest belong to the strip μ^i/λ.



μ^0/λ has no lower augmentable corner but has a unique upper augmentable one, namely, the lowest red cell in μ^1. μ^1/λ has a unique lower augmentable corner, the highest cell colored blue in μ^2. μ^2/λ is maximal.

5. Pushout of strips and row moves

Let (S, m) be an initial pair where $S = \mu/\lambda$ is a strip and $m = \nu/\lambda$ is a nonempty row move.

We say that (S, m) is compatible if it is reasonable, not contiguous, and is either (1) non-interfering, or (2) is interfering but is also pushout-perfectible; these notions are defined below. For compatible pairs (S, m) we define an output k-shape $\eta \in \Pi$ (see Subsections 5.4 and 5.5 for cases (1) and (2) respectively). This given, we define the pushout

$$\mathrm{push}(S, m) = (\tilde{S}, \tilde{m}) \tag{62}$$

which produces a final pair $(\tilde{S}, \tilde{m})$ where $\tilde{S} = \eta/\nu$ is a strip and $\tilde{m} = \eta/\mu$ is a move (possibly empty). This is depicted by the following diagram.

$$\begin{tikzcd} \lambda \arrow[r, "m"] \arrow[d, "S"'] & \nu \arrow[d, dotted, "\tilde{S}"] \\ \mu \arrow[r, dotted, "\tilde{m}"'] & \eta \end{tikzcd} \tag{63}$$

If S is a maximal strip then (S, m) is compatible (Proposition 125).

Property 111. *Let (S, m) be an initial pair. Then a modified column c of S cannot be a negatively modified column of m.*

Proof. Suppose otherwise. Let c be the leftmost modified column of $S = \mu/\lambda$ that is negatively modified by m. We have that c is also the leftmost negatively modified column of m since otherwise $\mathrm{cs}(\mu)$ would not be a partition. By the previous comment, $b = \mathrm{bot}_c(\partial\lambda)$ is leftmost in its row in $\partial\lambda$ and $h_\lambda(b) = k$. But looking at S we see that $h_\lambda(b) < k$, a contradiction. □

5.1. Reasonableness.
We say that the pair (S, m) is *reasonable* if for every string s of m, either $s \cap S = \emptyset$ or $s \subset S$. In other words, every string of m which intersects S must be contained in S.

Suppose the string s of m satisfies $s \subset S$. We say that S *matches* s *below* if $c_{s,d}$ is a modified column of S and otherwise say that S *continues below* s.

Lemma 112. *Suppose (S, m) is reasonable where $m = s_1 \cup s_2 \cup \cdots \cup s_r$. If S matches s_i below then S matches s_j below for each $j \leq i$. If S continues below s_i then S continues below s_j for each s_j on the same rows as s_i satisfying $j \leq i$.*

Proof. The first assertion follows directly from the assumption that $\mathrm{cs}(\mu)$ is a partition. In the case of the second assertion, we have that $\mathrm{bot}_c(\partial\lambda)$ belongs to the same row for every column c corresponding to such s_j's. Given that column c is not a modified column of S there is a cell of $\partial\lambda \setminus \partial\mu$ in column c. Given that μ/λ needs to be a skew diagram, the assertion follows. $\square$

Lemma 113. *Suppose (S, m) is reasonable. Then every modified column c of S which contains a cell in m is a positively modified column of m.*

Lemma 114. *Suppose (S, m) is reasonable and s is a string of m. Then $s \subset S$ if and only if S contains a cell in column $c_{s,u}$.*

Proof. The forward direction is immediate from Corollary 85. For the converse, suppose $c_{s,u}$ contains a cell x of S. Let d be the number of strings of m that are in the same rows as s and are equal to s or to its left. Then S contains the $d-1$ cells to the left of x. It follows from Property 111 that S contains d cells in the row of $\mathrm{bot}_{\mathrm{col}(x)}(\partial\lambda)$. This puts the top cell of s into S. By reasonableness $s \subset S$. $\square$

Proposition 115. *Suppose (S, m) is an initial pair with $S = \mu/\lambda$ maximal. Then (S, m) is reasonable.*

Proof. Let $s = \{a_1, a_2, \ldots, a_\ell\}$ be a string of m and suppose $a_i \in S$. As in Corollary 84, we choose the unique decomposition of S into cover-type strings such that the bottom cell of t_j is the j-th modified column of S for all j (going from left to right) and t_j is taken to be maximal given $t_1, \ldots, t_{j-1}$.

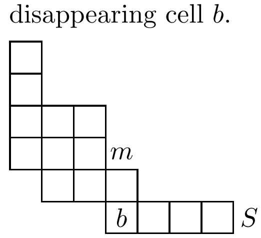
Suppose a_i is in the string t of S. It suffices to show that (1) $a_{i-1} \in t$ if $i > 1$ and (2) $a_{i+1} \in t$ if $i < \ell$. We prove (2) as (1) is similar.

The proof proceeds by induction on the indent $\mathrm{Ind}_m(s)$ of s in m. Suppose first that $\mathrm{Ind}_m(s) = 0$, that is, s is λ-addable. We have $a_{i+1} \in S$, for otherwise it would be a lower augmentable corner of S which would contradict the maximality of S by Proposition 103. By the choice of the decomposition of S, a_{i+1} and a_i are both in t.

Now suppose the Lemma holds for all strings s' of m with $\mathrm{Ind}_m(s') < d$. Let $s' = \{b_1, b_2, \ldots, b_\ell\}$ be the string of m preceding s. Since $d > 0$, b_j is just left of a_j for all j. Since $a_i \in S$ it follows that $b_i \in S$. By induction the cover-type string t' of S containing b_i contains s'. So $\mathrm{col}(b_i) = \mathrm{col}(a_i)^-$ is not a modified column of S. This implies that $\mathrm{col}(a_i)$ is also not a modified column of S. Due to the decomposition of S into covers, this means that t has a cell below a_i, that is, $a_{i+1} \in t$. $\square$

5.2. **Contiguity.** Suppose (S, m) is reasonable where $S = \mu/\lambda$ and m is a move from λ to ν. We say that (S, m) is *contiguous* if there is a cell $b \in \partial\mu \cap \partial\nu$ which is not present in $\partial(\mu \cup \nu)$; b is called a *disappearing cell.*

Example 116. The following strip and move (indicated by S and m respectively) are contiguous for $k = 6$ with disappearing cell b.



Lemma 117. *Suppose $b = (r, c)$ is a disappearing cell. Then*

1. *Column c is positively modified by m and contains no cells of S,*
2. *Row r is modified by S.*
3. *Column c contains an upper augmentable corner for S.*

Proof. Let n_m and n_S be respectively the number of cells of m and S in row r. If column c contains a cell of both S and m then we have the contradiction $h_{\mu\cup\nu}(b) = 1+h_\lambda(b)+\max(n_m, n_S) = \max(h_\mu(b), h_\nu(b)) \leq k$. A similar contradiction is reached if column c contains neither a cell of m nor one of S.

Suppose column c contains a cell in S; it is the cell x atop the column c of $\partial\lambda$. Then $n_m > n_S$ and $h_\nu(b) = k$ since b is a disappearing cell. Let y be the rightmost cell of m in row r, and observe that $y \notin S$. The move m removes the cell b^* just left of b and thus $\mathrm{cs}(\partial\lambda)_{c^-} = \mathrm{cs}(\partial\lambda)_c$. By Property 21 m cannot negatively modify column c^-. Therefore m has a cell x^* in column c^- just left of x that belongs to the same string of m as y. Since S is λ-addable, $x^* \in S$. But by reasonableness of (S, m), since $y \notin S$ we get the contradiction that $x^* \notin S$.

Therefore column c contains a cell of m (namely, x). We have $n_S > n_m$ and $h_\mu(b) = k$, and x has no cell of m contiguous to and below it. Item (1) follows.

If r is not a modified row of S then S removes the n_S cells just left of b and S contains n_S cells just left of x. Since m is λ-addable m also contains the n_S cells just left of x. But then m doesn't modify some of these columns (since $n_S > n_m$) while it modifies column c, contradicting Property 21. This proves (2).

It follows that $x \in m$ is an upper augmentable corner for S, proving (3). □

By Proposition 115, we have the following corollary.

Corollary 118. *Suppose S is a maximal strip and m a row move. Then (S, m) is non-contiguous.*

5.3. **Interference of strips and row moves.** Suppose that (S, m) is reasonable and non-contiguous. Then

$$\mathrm{cs}(\mu) - \mathrm{cs}(\lambda) + \mathrm{cs}(m * \lambda) = \Delta_{\mathrm{cs}}(S) + \Delta_{\mathrm{cs}}(m) + \mathrm{cs}(\lambda).$$

Recalling Notation 39 let

$$m' = \bigcup \{\text{ strings } s \subset m \mid s \text{ and } S \text{ are not matched below}\} \tag{64}$$

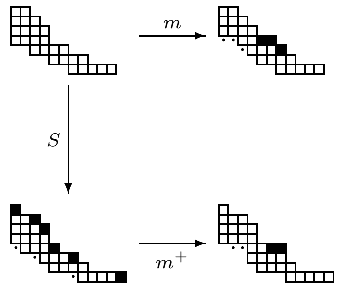
$$m^+ = \uparrow_S (m'). \tag{65}$$

Define the vector $\Delta_{\mathrm{cs}}(m')$ by considering only the modified columns of strings in m'. We say that (S, m) is *non-interfering* if $\mathrm{cs}(\lambda) + \Delta_{\mathrm{cs}}(S) + \Delta_{\mathrm{cs}}(m')$ is a partition, and *interfering* otherwise.

Remark 119. (S, m) is interfering if and only if S and the last string s of m are not matched below, S modifies column c^+, and $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c^+} + 1$, where $c = c_{s,u}$.

Example 120. With $k = 7$ the pair (S, m) is interfering: there is a violation of the k-shape property in $m^+ * \mu$. The set of cells m' is comprised of the second and third strings of m. In passing from m' to m^+ the third string has been bumped up

a row.



Lemma 121. *The set of cells m^+ satisfies all the conditions for a move from μ except that $(m^+) * \mu$ may not be a k-shape. Furthermore, we have $\mathrm{cs}((m^+) * \mu) = \mathrm{cs}(\lambda) + \Delta_{\mathrm{cs}}(S) + \Delta_{\mathrm{cs}}(m')$.*

Proof. Let $m^+ = t_1 \cup t_2 \cup \cdots \cup t_\rho$ where each t_i is either a string in $m' \setminus S$, or a string in $m' \cap S$ shifted upwards. We assume that the t_i are ordered from left to right, as is the convention for row moves. It is clear that the t_i are weak translates of each other in the correct columns. In order to prove the lemma, we will show that they are successive row type addable strings that are translates of the strings of m. We proceed by induction on i.

First suppose that $t_i \subset m'$ was not bumped up. By Lemma 114, $c_{s,u}$ contains no cells of S. By non-contiguity and Corollary 85, column $c_{s,d}$ is identical in λ and μ, and also in ν and $\mu \cup \nu$. Thus t_i is a row type string of $\mu \cup t_1 \cup \cdots \cup t_{i-1}$ equal to s_i.

Now suppose that t_i was bumped up from $s_i \in m' \cap S$. By Lemma 112, it suffices to check the case that s_i is λ-addable. First we show that t_i is addable. This is clear if s_i is not equal to s_1, for s_{i-1} is higher than s_i. Lemma 122 deals with the case $s_i = s_1$. The diagram of t_i is a translate of that of s_i by Lemma 114 and the assumption that S continues below s_i, which ensures that the their modified columns agree in size. □

Lemma 122. *Suppose S continues below the first string $s_1 = \{a_1, a_2, \ldots, a_\ell\}$ of m. For each $i \in [1, \ell]$ let c_i be the column containing a_i. Then there is an addable corner of μ in column c_i.*

Proof. Consider the case $i = \ell$ and set $c = c_\ell$. We prove the equivalent statement that column c^- either intersects S or satisfies $(\lambda)_{c^-} \geq (\lambda)_c + 2$. Since m is a move, we have $\mathrm{cs}(\lambda)_{c^-} > \mathrm{cs}(\lambda)_c$. Assume there is no cell of S in column c^-. Then Corollary 85 and "continuing below" imply that the bottom of c^- in $\partial\lambda$ starts higher than that of c. This implies $(\lambda)_{c^-} \geq (\lambda)_c + 2$.

The general case then follows from Lemma 86 since s_1 is λ-addable and there is a μ-addable corner in column $c = c_\ell$. □

5.4. **Row-type pushout: non-interfering case.** Let (S, m) be reasonable, non-contiguous, and non-interfering. Then by definition we declare (S, m) to be compatible, set $\eta = (m^+) * \mu$, let $(\tilde{S}, \tilde{m})$ be as in (63), and define the pushout of (S, m) by (62). By Lemma 121 and Proposition 123, $\tilde{m}$ is a (possibly empty) row move and $\tilde{S}$ is a strip.

Proposition 123. *Suppose (S, m) is non-interfering. Then η/ν is a strip.*

Proof. It is immediate that η/ν is a horizontal strip. We have $\mathrm{rs}(\eta)/\mathrm{rs}(\nu) = \mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$, which is a horizontal strip by assumption. Also

$$\begin{aligned}\mathrm{cs}(\eta) - \mathrm{cs}(\nu) &= \Delta_{\mathrm{cs}}(S) + \Delta_{\mathrm{cs}}(m') - \Delta_{\mathrm{cs}}(m) \\ &= \Delta_{\mathrm{cs}}(S) - \Delta_{\mathrm{cs}}(m \setminus m')\end{aligned}$$

Observe that $m \backslash m'$ corresponds to the strings s of m such that s and S are matched below. Therefore $\mathrm{cs}(\eta) - \mathrm{cs}(\nu)$ is a 0-1 vector since the positively modified columns of $m \setminus m'$ cancel out with some modified columns of S, and the negatively modified columns of $m \setminus m'$ do not coincide with modified columns of S by Property 111. $\square$

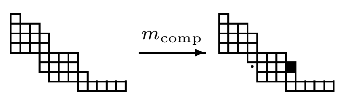
5.5. **Row-type pushout: interfering case.** Suppose (S, m) is reasonable, non-contiguous, and interfering. We say that (S, m) is *pushout-perfectible* if there is a set of cells m_{comp} outside $(m^+) * \mu$ such that if

$$\eta = ((m^+) * \mu) \cup m_{\mathrm{comp}} \tag{66}$$

then η/ν is a strip and η/μ is a row move from μ with the same initial string as m^+. By Proposition 27, m_{comp} is unique if it exists.

In the case that (S, m) is pushout-perfectible, then by definition we declare (S, m) to be compatible. With η as in (66) we define $(\tilde{S}, \tilde{m})$ and the pushout of (S, m) by (63) and (62). By definition $\tilde{S}$ is a strip and $\tilde{m}$ is a row move.

Example 124. Continuing Example 120, the cells of m_{comp} are darkened as added to $m^+ * \mu$:



Proposition 125. *Suppose (S, m) is interfering with m a row move and S maximal. Then (S, m) is pushout-perfectible (and hence compatible). Furthermore the strings of m_{comp} lie on the same rows as the final string of m and no column contains both cells of m_{comp} and S.*

Proof. (S, m) is reasonable and non-contiguous by Proposition 115 and Corollary 118, so it makes sense to consider interference.

By Remark 119, (S, m) interferes only if there is a modified column c of S such that column c^- is the rightmost negatively modified column of m and $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c^-} - 1$. Let $b = (r, c) = \mathrm{bot}_c(\partial\lambda)$ and $b' = \mathrm{bot}_{c^-}(\partial\lambda)$. We have $h_\mu(b) \le k$ since c is a modified column of S. If $\mathrm{row}(b) < \mathrm{row}(b')$ then $h_\mu(b) \ge h_\lambda(b') = k$ and thus $h_\mu(b) = k$. This means that S has a lower augmentable corner in row r, contradicting Proposition 103 and maximality. Therefore b and b' are in row r, and this row corresponds to the row of the top cell of the last string of m. Now suppose that c^+ is also a modified column of S with $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c^+}$, and let $\bar{b} = \mathrm{bot}_{c^+}(\partial\lambda)$. By the same argument we get that b and $\bar{b}$ lie in the same row. Continuing in this way, we get that all modified columns d of S such that $\mathrm{cs}(\lambda)_d = \mathrm{cs}(\lambda)_c$ occupy the same rows. If there are ℓ of them and ℓ' cells of m in row r, we have established that $\lambda_{r^-} - \lambda_r \ge \ell + \ell'$. Therefore $\rho = m^+ * \mu$ is such that $\rho_{r^-} - \rho_r \ge \ell$ since exactly ℓ' cells of $m^+ \cup S$ lie in that row by hypothesis (otherwise column c would not be a modified column of S). By Lemma 93, any row R below row r that contains a cell of the last string of m is also such that $\lambda_{R^-} - \lambda_R \ge \ell + \ell'$. Furthermore, for any such row R we also have $\rho_{R^-} - \rho_R \ge \ell$ since again exactly ℓ' cells of $m^+ \cup S$ lie

in that row by hypothesis (otherwise there would be an upper augmentable corner associated to a given row R, contradicting maximality).

We have established that ℓ cells can be added to the right of every cell of the last string of m in ρ, and from our proof, these cells do not lie above cells of S. Let m_{comp} be the union of those cells. Defining $\eta = \rho \cup m_{\text{comp}}$, it is clear that η/ν is a horizontal strip. We have $\text{rs}(\eta) = \text{rs}(\mu)$ so $\text{rs}(\eta)/\text{rs}(\nu)$ is a horizontal strip. Finally, one checks that $\text{cs}(\eta)$ is a partition and $\text{cs}(\eta)/\text{cs}(\nu)$ a horizontal strip in the same manner as in Proposition 123. $\square$

5.6. **Alternative description of pushouts (row moves).** Suppose $m = s_1 \cup s_2 \cup \cdots$ is a row move such that $\Delta_{\text{cs}}(s_1)$ affects columns c and $c+d$. If α is not a partition, we suppose that $\alpha_i + 1 = \alpha_{i+1} = \alpha_{i+2} = \cdots = \alpha_{i+a} > \alpha_{i+a+1}$. Then the perfection of α with respect to m is the vector

$$\text{per}_m(\alpha) = \begin{cases} \alpha + \sum_{j=1}^{a} (e_{i+j+d} - e_{i+j}) & \text{if } \alpha \text{ is not a partition} \\ \alpha & \text{if } \alpha \text{ is a partition} \end{cases}$$

Here e_j denotes the unit vector with a 1 in the j-th position and 0's elsewhere.

Let $(S = \mu/\lambda, m = \nu/\lambda)$ be any initial pair where $m = s_1 \cup \cdots \cup s_r$. Let m' be the collection of cells obtained from m by removing s_i whenever the positively modified column of s_i is a modified column of S. It is easy to see that m' is of the form $s_j \cup s_{j+1} \cup \cdots \cup s_r$. The *expected column shape* $\text{ecs}(S, m)$ of (S, m) is defined to be

$$\text{ecs}(S, m) = \text{per}_m(\text{cs}(\lambda) + \Delta_{\text{cs}}(S) + \Delta_{\text{cs}}(m')).$$

Proposition 126. *Let $(S = \mu/\lambda, m = \nu/\lambda)$ be an initial pair where $m \neq \emptyset$ is a row move. Suppose there exists a k-shape η so that*

(1) $\text{cs}(\eta) = \text{ecs}(S, m)$
(2) *η/μ is either empty or a row-move whose string diagrams are translates of those of m*
(3) $\nu \subset \eta$.

Then (S, m) is compatible and $\text{push}(S, m) = (\eta/\nu, \eta/\mu)$. *In particular, (η/ν) is a strip.*

Proof. It is easy to see that η/μ decomposes into row type strings as $m'' \cup m_{\text{comp}}$ where $\text{cs}(m'' * \mu) = \text{cs}(\mu) + \Delta_{\text{cs}}(m')$. Since m'' modifies the same columns as m', and the two have the same diagrams we conclude that each string of m'' is either a string in m' or a string in m' shifted up one cell. But m'' is a collection of strings on μ, so the strings in m' must be reasonable with respect to S.

We now claim that $m_{\text{comp}} \cap m = \emptyset$. Suppose otherwise. Let a be the rightmost cell in the intersection $m_{\text{comp}} \cap m$, lying in a string $s \in (m \setminus m')$ and a string $t \in m_{\text{comp}}$. If a is not the rightmost cell in s we let b be the cell immediately right of a in s. Now s and t have the same diagram so we deduce that the cell b' after a in t is either equal to b or immediately to the left of b. In either case, this contradicts the assumption that a is rightmost. Thus a is in the positively modified column c of s. But by the original assumptions c is also a modified column of S. This contradicts the fact that $m_{\text{comp}} \cap S = \emptyset$ and we conclude $m_{\text{comp}} \cap m = \emptyset$. Now we apply (3) to see that all strings $m \setminus m'$ must have already been contained in μ – thus (S, m) is reasonable.

Suppose (S, m) is contiguous. By Lemma 117, this means there is a disappearing cell b, and b is in a column c which does not contain cells of S but does contain cells

of m (it is in fact a positively modified column of m). By reasonableness, the column c thus contains cells of m' and in particular is not in a modified column of m_{comp}. Thus $\mathrm{cs}(\lambda)_c = \mathrm{ecs}(S,m)_c - 1$. However, the disappearance implies $\mathrm{cs}(\eta)_c = \mathrm{cs}(\lambda)_c$, a contradiction.

To show that η/ν is a horizontal strip, we only need to show that no cell of m_{comp} lies above a cell of S. Suppose x is the leftmost cell in m_{comp} that lies above a cell of S, and let r be the row of x. Let $s \in m_{\mathrm{comp}}$ be the string that contains x and let c be the column of the cell removed in row r when adding s. Since $\mathrm{rs}_r(\mu) \leq \mathrm{rs}_{r^-}(\lambda)$ we have that c is weakly to the right of $\mathrm{left}_{r^-}(\partial\lambda)$ and thus the cell in row r^- and column c belongs to $\partial\lambda \setminus \partial\mu$. Hence, by Corollary 85, there is a cell of S in column c. First assume that x is not the highest cell in s, and let y be above x in s. Then y is in column c and we either have the contradiction that y lies above a cell of S or that $m_{\mathrm{comp}} \cap S \neq \emptyset$. Now assume that x is the highest cell in its string. This time we have the contradiction that c is not a modified column of S.

Since $\mathrm{rs}(\eta) = \mathrm{rs}(\mu)$ and $\mathrm{rs}(\nu) = \mathrm{rs}(\lambda)$ we have that $\mathrm{rs}(\eta)/\mathrm{rs}(\nu)$ is a horizontal strip. Finally, by supposition the negatively modified columns of m_{comp} are positively modified columns of S and the lowest cell of each string of m_{comp} modifies positively its column. Since η/ν is a horizontal strip, we have that $\mathrm{cs}(\eta)/\mathrm{cs}(\nu)$ is a vertical strip. $\square$

Lemma 127. *Let (S, m) be a compatible initial pair with m a row move. Then the set of strings m_{comp} and the row move m do not share any columns.*

Proof. In the proof of Proposition 126 it was shown that no cell of m_{comp} can lie above a cell of S. Therefore the cells of m_{comp} lie above cells of λ. Suppose m_{comp} and m share columns. Then they intersect, and must do so in $m \setminus m' \subset S$, a contradiction. $\square$

6. Pushout of strips and column moves

In this section we consider initial pairs $(S = \mu/\lambda, m = \nu/\lambda)$ consisting of a strip and a column move.

We define (S, m) to be compatible if it is reasonable, non-contiguous, normal, and either (1) it is non-interfering or (2) it is interfering but is pushout-perfectible; these notions are defined below. As for row moves, in each of the above cases we specify an output k-shape $\eta \in \Pi$ and define the pushout of (S, m) and the final pair $(\tilde{S}, \tilde{m})$ as in (62) (63).

We omit proofs which are essentially the same in the row and column cases.

6.1. **Reasonableness.** We say that the pair (S, m) is *reasonable* if for every string $s \subset m$, either $s \cap S = \emptyset$, or $s \subset S$. If $s \subset m$ is contained inside S, we say that S *matches s above* if $r_{s,u}$ is a modified row of S. Otherwise we say that S *continues above s*.

Lemma 128. *Let (S, m) be any initial pair. If a modified row of S contains a cell of m, then that row intersects the initial string of m. If a modified row of S is a negatively modified row of m, then S intersects the initial string $s \subset m$. In particular, if (S, m) is reasonable, only the initial string $s \subset m$ can be matched above.*

Proof. Follows immediately from the definition of column moves and the fact that $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. $\square$

Lemma 129. *Suppose (S, m) is reasonable. Then every modified row r of S which contains a cell in m is a positively modified row of m.*

Lemma 130. *Suppose (S, m) is reasonable. If $s \not\subseteq S$ then S does not contain a cell in row $r_{s,d}$.*

Proposition 131. *Let S be a maximal strip and m a column move. Then (S, m) is reasonable.*

6.2. **Normality.** Let $s \subset m$ be the initial string of m. We say that (S, m) is *normal*, if it is reasonable and in the case that S continues above s then (a) none of the modified rows of S contain cells of s (and by Lemma 128, none of the modified rows of S contain cells of m) and (b) the negatively modified row of s is not a modified row of S.

Proposition 132. *Let S be a maximal strip and m any column move. Then (S, m) is normal.*

Proof. The pair (S, m) is reasonable by Proposition 131. Suppose S continues above $s = \{a_1, a_2, \dots, a_\ell\}$, where the cells are indexed by decreasing diagonal index. By Lemma 128, normality cannot be violated if s is not the initial string of m, so we suppose s is the initial string of m. Since S does not match s above by definition, the row r_ℓ containing a_ℓ is not a modified row of S. The claim is thus trivial if $\ell = 1$ so we assume $\ell > 1$. Suppose r_ℓ contains $p \geq 1$ cells of S, implying that the p leftmost cells of r_ℓ are moved when going from $\partial\lambda$ to $\partial\mu$ (and none of the columns of these p cells are modified columns of S). It follows from Property 82 and $\mathrm{rs}(\lambda)_{r_\ell} < \mathrm{rs}(\lambda)_{r_\ell^-}$ that $\lambda_{r_\ell^-} \geq \lambda_{r_\ell} + p + 1$. In particular, there is an addable corner b^* on row r_ℓ of μ.

It is easy to see that the row $r_{\ell-1}$ containing $a_{\ell-1}$ contains at least p cells of S, with equality if and only if $r_{\ell-1}$ is not a modified row of S. If $r_{\ell-1}$ is a modified row of S, then the addable corner b^* will be an upper augmentable corner for S, contradicting maximality and Proposition 103. So $r_{\ell-1}$ is not a modified row of S.

Since $\lambda_{r_\ell^-} \geq \lambda_{r_\ell} + p + 1$, we get by Lemma 31 that $\lambda_{r_{\ell-1}^-} \geq \lambda_{r_{\ell-1}} + p + 1$, so that row $r_{\ell-1}$ of μ also has an addable corner. Continuing as before we see that (S, m) is normal. □

Lemma 133. *Suppose (S, m) is normal and let $s \subset m$ be any string such that S continues above s. Then S contains the same number of cells in each row r containing a cell of s, and also the same number of cells in the negatively modified row of s. Furthermore, if s is the initial string of m, then each such row r has a μ-addable corner.*

Proof. The first statement follows easily from the definition of normality. The last statement is proven as in Proposition 132. □

6.3. **Contiguity.** Suppose (S, m) is reasonable. We say that (S, m) is contiguous if there is a cell $b \in \partial\mu \cap \partial\nu$ which is not present in $\partial(\mu \cup \nu)$. Call such a b a *disappearing cell*.

Lemma 134. *Suppose $b = (r, c)$ is a disappearing cell. Then*

(1) *Row r is a positively modified row of m and does not contain cells of S,*
(2) *Column c is a modified column of S.*
(3) *Row r contains a lower augmentable corner for S.*

Proof. Suppose row r contains $p \geq 1$ cells of S. Then m must contain cells in column c. Let b' be the cell below $\mathrm{bot}_c(\partial\nu)$, $R = \mathrm{row}(b')$, and $h = \mathrm{cs}(\nu)_c$. We have the sequence of inequalities:

$$\begin{aligned}\mathrm{rs}(\lambda)_R - \mathrm{rs}(\mu)_r &= (h + \mathrm{rs}(\lambda)_R) - (h - 1 + \mathrm{rs}(\mu)_r) - 1 \\ &\leq k + 1 - (h - 1 + \mathrm{rs}(\mu)_r) - 1 \\ &\leq k + 1 - h_{\nu\cup\mu}(b) - 1 \\ &\leq -1\end{aligned}$$

which contradicts the fact that $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip. Thus row r contains no cells of S and contains exactly one cell $x \in m$. Also column c exactly one cell $a \in S$ and no cells of m. Thus $h_\lambda(b) = k - 1$.

Suppose r is not a positively modified row of m. Then m contains a cell a^* in the column of the cell b^* immediately left of b, and we have $h_\lambda(b^*) = k$. But a^* is in the same row as a, so $a^* \in S$ as well. But by reasonableness, $x \in S$, a contradiction. This proves (1).

Suppose c is not a modified column of S. Then there is a cell $x' \in S$ contiguous to and below $a \in S$. Considering hook lengths we conclude that x' is just below x, S removes the cell b' just below b, $h_\lambda(b') = k$, and $\lambda_r = \lambda_{r^-}$. But since m is λ-addable it follows that $x' \in m$. But then row r^- must be positively modified by m, contradicting $h_\lambda(b') = k$. This proves (2) and that x is λ-addable.

For (3), the cell x is a lower augmentable corner for S. □

Corollary 135. *Suppose S is a maximal strip and m any move. Then (S, m) is not contiguous.*

6.4. **Interference of strips and column moves.** Suppose that (S, m) is normal and non-contiguous. Define $\Delta_{\mathrm{rs}}(S) = \mathrm{rs}(\mu) - \mathrm{rs}(\lambda)$ and $\Delta_{\mathrm{rs}}(m) = \mathrm{rs}(m * \lambda) - \mathrm{rs}(\lambda)$. Similarly define $\Delta_{\mathrm{rs}}(s)$ for a column-type string s. Thus

$$\mathrm{rs}(\mu) - \mathrm{rs}(\lambda) + \mathrm{rs}(m * \lambda) = \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m) + \mathrm{rs}(\lambda).$$

Recalling Notation 39 let

$$m' = \{\text{strings } s \subset m \mid s \text{ and } S \text{ are not matched above}\} \subset m \tag{67}$$

$$m^+ = \rightarrow_S (m'). \tag{68}$$

By Lemma 128, the set m' is obtained from m by possibly removing the initial string of m. Define the vector $\Delta_{\mathrm{rs}}(m')$ by considering only the modified rows of strings inside m'.

If $m' \neq \emptyset$ we say that (S, m) is *non-interfering* if $\mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')$ is a partition and *interfering* otherwise. If $m' = \emptyset$ we say that (S, m) is non-interfering if $\mathrm{rs}(\mu)/\mathrm{rs}(\nu)$ is a horizontal strip and interfering otherwise (observe that $\mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta(m') = \mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) = \mathrm{rs}(\mu)$ is always a partition in that case). The latter case is referred to as *special interference.*

Lemma 136. *The set of cells m^+ satisfies all the conditions for a move on μ except that $(m^+) * \mu$ may not be a k-shape. Furthermore, we have $\mathrm{rs}((m^+) * \mu) = \mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')$. In particular, $(m^+) * \mu$ is always a k-shape when (S, m) is non-interfering.*

Proof. The proof is similar to Lemma 121, except that we now use Lemmata 133 and 137. □

Lemma 137. *Suppose S continues above the first string $s_1 = \{a_1, a_2, \dots, a_\ell\}$ of m. For each $i \in [1, \ell]$ let r_i be the row containing a_i. Then there is an addable corner of μ in row r_i. Moreover, the addable corner of μ in row r_i does not lie above a cell of S.*

Proof. Consider the case $i = \ell$ and set $r = r_\ell$. Since S does not match s_1 above by definition, the row r_ℓ containing a_ℓ is not a modified row of S by normality. Suppose r_ℓ contains $p \geq 1$ cells of S, implying that the p leftmost cells of r_ℓ are moved when going from $\partial\lambda$ to $\partial\mu$ (and none of the columns of these p cells are modified columns of S). It follows from Property 82 and $\mathrm{rs}(\lambda)_{r_\ell} < \mathrm{rs}(\lambda)_{r_\ell^-}$ that $\lambda_{r_\ell^-} \geq \lambda_{r_\ell} + p + 1$. In particular, there is an addable corner in row r_ℓ of μ and it does not lie above a cell of S.

Since $s_1 = \{a_1, a_2, \dots, a_\ell\}$ is λ-addable, Lemma 31 ensures that $\lambda_{r_i^-} \geq \lambda_{r_i} + p + 1$ for all i. There are exactly p cells of S in row r_i for all i by Lemma 133. So again there is an addable corner in row r_i of μ and it does not lie above a cell of S. □

6.5. **Column-type pushout: non-interfering case.** Suppose (S, m) is normal, non-contiguous, and non-interfering. In this case, by definition (S, m) is declared to be compatible where we set $\eta = (m^+) * \mu$ and define $(\tilde{S}, \tilde{m})$ by (63) and the pushout of (S, m) by (62). $\tilde{m}$ is a (possibly empty) column move and $\tilde{S}$ is a strip by Lemma 136 and Proposition 138.

Proposition 138. *Suppose (S, m) is normal, non-contiguous, and non-interfering. Then η/ν is a strip.*

Proof. That η/ν is a horizontal strip is not difficult (Lemma 137 ensures that the cells of m^+ do not lie above cells of S). We also have $\mathrm{cs}(\eta)/\mathrm{cs}(\nu) = \mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$.

If $m' = \emptyset$ we have by definition that if (S, m) is non-interfering then $\mathrm{rs}(\eta)/\mathrm{rs}(\nu) = \mathrm{rs}(\mu)/\mathrm{rs}(\nu)$ is a horizontal strip. Thus η/ν is a strip.

Suppose m' is not empty and that $\mathrm{rs}(\eta) = \mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')$ is a partition. We must prove that $\mathrm{rs}(\nu)_{r^-} \geq \mathrm{rs}(\eta)_r \geq \mathrm{rs}(\nu)_r$ for each row r. Recall that modified rows of m' are modified rows of m and that the only string that may possibly be in $m \setminus m'$ is the initial one. Therefore the second inequality follows from the fact that if $m \setminus m'$ is not empty then the positively modified row of the initial string of m is a modified row of S.

To prove the first inequality, observe that

$$\mathrm{rs}(\nu)_{r^-} - \mathrm{rs}(\eta)_r = \mathrm{rs}(\lambda)_{r^-} - \mathrm{rs}(\mu)_r + \Delta_{\mathrm{rs}}(m)_{r^-} - \Delta_{\mathrm{rs}}(m')_r \,,$$

with $\mathrm{rs}(\lambda)_{r^-} - \mathrm{rs}(\mu)_r \geq 0$ since S is a strip.

Suppose that $m = m'$. Then the first inequality can only fail if r^- is the uppermost negatively modified row of m or if r is the positively modified row of the initial string of m.

Suppose that m is non-degenerate. Let r^- be the uppermost negatively modified row of m. By normality we have $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_{r^-}$. Therefore if the first inequality fails, we have $\mathrm{rs}_{r^-}(\mu) = \mathrm{rs}_{r^-}(\lambda) = \mathrm{rs}_r(\mu)$ which is a contradiction since η would not then be a k-shape (r^- is a negatively modified row of m' that is not a modified row of S by normality). Let r be the positively modified row of the initial string of m. By normality we have $\mathrm{rs}(\lambda)_r = \mathrm{rs}(\mu)_r$. Therefore if the first inequality fails, we have $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_r = \mathrm{rs}(\lambda)_r$ which is a contradiction since λ would not then be a k-shape (r^- is a negatively modified row of m).

Suppose that m is degenerate, and let r^- be the uppermost negatively modified row of m (and thus r is the positively modified row of the initial string of m). By normality we have $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_{r^-}$ and $\mathrm{rs}(\lambda)_r = \mathrm{rs}(\mu)_r$. Therefore if the first inequality fails, we have $\mathrm{rs}(\lambda)_{r^-} \le \mathrm{rs}(\mu)_r + 1 = \mathrm{rs}(\lambda)_r + 1$ which is a contradiction since λ would not then be a k-shape ($\mathrm{rs}(\lambda)_{r^-} - \mathrm{rs}(\lambda)_r \ge 2$ since m is degenerate).

Finally, suppose that m and m' are distinct. The only case to consider that was not considered in the case $m = m'$ is when r^- is the negatively modified row of the first string of m. By hypothesis m' is not empty and so r is also a negatively modified row of m'. The first inequality then follows immediately. □

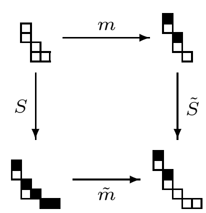
6.6. **Column-type pushout: interfering case.** Suppose (S, m) is normal, non-contiguous, and interfering. We say that (S, m) is *pushout-perfectible* if there is a set of cells m_{comp} outside $(m^+) * \mu$ so that if

$$\eta = ((m^+) * \mu) \cup m_{\mathrm{comp}} \tag{69}$$

then η/ν is a strip and η/μ is a column move from μ whose strings have the same diagram as those of m. Since $\mathrm{rs}(\eta)/\mathrm{rs}(\nu)$ is a horizontal strip, m_{comp} can only be a single column-type string and will thus be unique if it exists.

In the case that (S, m) is pushout-perfectible, by definition we declare (S, m) to be compatible where η is specified by (69) and define $(\tilde{S}, \tilde{m})$ by (63) and the pushout of (S, m) by (62). By definition, $\tilde{m}$ is a column move and $\tilde{S}$ is a strip.

Example 139. This is an example of special interference for $k = 3$. In $\mu = S * \lambda$ and $\nu = m * \lambda$ the new cells added to λ are shaded. In the lower right k-shape the cells of m_{comp} are shaded.



Lemma 140. *Let (S, m) be such that m' is empty. Then there is (special) interference iff $\mathrm{rs}(\mu)_{r^-} = \mathrm{rs}(\mu)_r$, where r^- is the negatively modified row of m (m is necessarily of rank 1).*

Proof. Let $m' = \emptyset$. In this case there is interference iff $\mathrm{rs}(\mu)/\mathrm{rs}(\nu)$ is not a horizontal strip. We have

$$\mathrm{rs}(\nu)_{r^-} - \mathrm{rs}(\mu)_r = \mathrm{rs}(\lambda)_{r^-} - \mathrm{rs}(\mu)_r + \Delta_{\mathrm{rs}}(m)_{r^-} ,$$

with $\mathrm{rs}(\lambda)_{r^-} - \mathrm{rs}(\mu)_r \ge 0$ since S is a strip. The inequality $\mathrm{rs}(\nu)_{r^-} - \mathrm{rs}(\mu)_r \ge 0$ can thus only fail when r^- is the negatively modified row of m. In that case, by normality we have $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_{r^-}$. Therefore we obtain

$$\mathrm{rs}(\nu)_{r^-} - \mathrm{rs}(\mu)_r = \mathrm{rs}(\mu)_{r^-} - \mathrm{rs}(\mu)_r - 1 < 0 \iff \mathrm{rs}(\mu)_{r^-} = \mathrm{rs}(\mu)_r$$

and the lemma follows. □

Proposition 141. *Suppose (S, m) is interfering, S is maximal, and m is a column move. Then (S, m) is pushout-perfectible (and hence compatible). Moreover m_{comp} consists of a single string lying in the same columns as the last string of m.*

Proof. By Proposition 132 and Corollary 135, (S, m) is normal and not contiguous, so that it makes sense to refer to interference.

Suppose m^+ is non-empty so that $\mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')$ is not a partition. Let r^- be the negatively modified row of the final string s of m. We may assume that r^- is not a modified row of S, for otherwise s would have to be initial, and Lemma 130 would imply that $s \subseteq S$ and thus that s is matched above, implying that (S, m) does not interfere. Also, r must be a modified row of S for interference to occur. Note that since $\mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')$ is not a partition we have $\mathrm{rs}(\mu)_{r^-} = \mathrm{rs}(\mu)_r$ and thus $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_r$ (r^- is not a modified row of S).

We claim that η has an addable corner directly above the first cell a of s. Since $h_\lambda(\mathrm{left}_{r^-}(\partial\lambda)) = k$ by the definition of a move, we have from $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_r$ that $h = h_\mu(\mathrm{left}_r(\partial\mu))$ is $k-1$ or k. In either case (using Lemma 101(3) when $h = k$) we see that $\mathrm{left}_{r^-}(\partial\lambda)$ lies in the same column as $\mathrm{left}_r(\partial\mu)$. We then have immediately that there is an addable corner directly above the first cell a of s. Since s is λ-addable, we obtain from Lemma 86 that there is a μ-addable corner above every cell of m. The rest of the proof that $m^+ \cup m_{\mathrm{comp}}$ is a column move is analogous to the proof of Lemma 102.

Suppose m^+ is empty and $\mathrm{rs}(\eta)/\mathrm{rs}(\nu) = \mathrm{rs}(\mu)/\mathrm{rs}(\nu)$ is not a horizontal strip. Recall that only the initial string of m can disappear and thus m is of rank 1. From Lemma 140, the negatively modified row of m is in a row r^- such that row r is a modified row of S with $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_{r^-} = \mathrm{rs}(\mu)_r$ (recall that $\mathrm{rs}(\lambda)_{r^-} = \mathrm{rs}(\mu)_{r^-}$ by normality). Again η has an addable corner directly above the first cell a of s by Lemma 101(3). The rest of the proof that m_{comp} is a column move is as in the non-empty case.

Since the cells of m_{comp} lie above cells of m we have that η/ν is a horizontal strip. Obviously $\mathrm{cs}(\eta)/\mathrm{cs}(\nu) = \mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$ is a vertical strip. We thus only have to prove that $\mathrm{rs}(\eta)/\mathrm{rs}(\nu)$ is a horizontal strip. If m' is non-empty there is interference only if $\mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')$ is not a partition. Following the proof of Proposition 138 we have that $\mathrm{rs}(\nu)_i \geq \mathrm{rs}(\eta)_{i+1}$ for all i except possibly when $i = R$ is the highest positively modified row of m. In that case $\mathrm{rs}(\eta)/\mathrm{rs}(\nu)$ is a horizontal strip since the positively modified row R^+ of m_{comp} lies in the row above row R and $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip by definition (that is, given $\lambda_R \geq \mu_{R^+}$, $\eta_{R^+} = \mu_{R^+} + 1$ and $\nu_R = \lambda_R + 1$, we have $\nu_R \geq \eta_{R^+}$). If m' is empty, then by Lemma 140 there is interference iff $\mathrm{rs}(\mu)_i = \mathrm{rs}(\mu)_{i+1}$, where $i = r^-$ is the negatively modified row of $m = s$. In that case, given $\eta_r = \mu_r - 1$ and $\nu_{r^-} = \lambda_{r^-} - 1$, the fact that $\lambda_{r^-} \geq \mu_r$ guarantees that $\nu_{r^-} \geq \eta_r$. So we only have to check what happens at the positively modified row of m_{comp}. The result follows just as in the $m' \neq \emptyset$ case. □

Lemma 142. *Suppose (S, m) is pushout-perfectible. If any cell of the string $s = m_{\mathrm{comp}}$ lies above a cell of S then s lies in the same columns as the final string of m.*

Proof. Let a be a cell of s that lies above a cell of S. From the definition of $s = m_{\mathrm{comp}}$, there is a cell of the final string t of m in the row below that of a. Hence, since S is a horizontal strip, a also lies above a cell of t. Finally, since s and t are translates the lemma follows. □

Lemma 143. *Suppose (S, m) is pushout-perfectible. If the string $s = m_{\mathrm{comp}}$ does not lie in the same column as the last string of m then the first cell a of s is an upper augmentable corner of S.*

Proof. By the definition of interference, S modifies the row r above the highest negatively modified row of m. From the hypotheses, $h_\mu(r, \mathrm{col}(a)) = k$, so that the cell a is contiguous and above a cell of S in row r. By Lemma 142 a does not lie above a cell of S and is therefore an upper augmentable corner of S. □

6.7. **Alternative description of pushouts (column moves).** Let $(S, m) = (\mu/\lambda, \nu/\lambda)$ be any initial pair where $m = s_1 \cup \cdots \cup s_r$ is a column move. Let m' be the collection of cells obtained from m by removing s_i whenever the positively modified row of s_i is a modified row of S. It is easy to see that m' is of the form $s_1 \cup s_2 \cup \cdots \cup s_r$ or $s_2 \cup \cdots \cup s_r$. Suppose that $\Delta(s_1)$ affects rows c and $c + d$. If α is not a partition, we suppose that $\alpha_i + 1 = \alpha_{i+1} > \alpha_{i+2}$. We say that there is interference if α is not a partition or if m' is empty and $\alpha_i = \alpha_{i+1}$, where $i = c$ is the negatively modified row of s_1. Then the perfection of α with respect to (S, m) is the vector

$$\mathrm{per}_{S,m}(\alpha) = \begin{cases} \alpha + e_{i+d+1} - e_{i+1} & \text{if there is interference} \\ \alpha & \text{otherwise} \end{cases}$$

The *expected row shape* $\mathrm{ers}(S, m)$ of (S, m) is defined to be

$$\mathrm{ers}(S, m) = \mathrm{per}_{S,m}(\mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S) + \Delta_{\mathrm{rs}}(m')).$$

Proposition 144. *Let $(S = \mu/\lambda, m = \nu/\lambda)$ be an initial pair with m a nonempty column move. Suppose there exists a k-shape η such that*

(1) $\mathrm{rs}(\eta) = \mathrm{ers}(S, m)$.
(2) *η/μ is either empty or a column move whose strings are translates of those of m.*
(3) $\nu \subset \eta$.

Then (S, m) is compatible and $\mathrm{push}(S, m) = (\eta/\nu, \eta/\mu)$. *In particular η/ν is a strip.*

Proof. It is easy to see that η/μ decomposes into column type strings as $m'' \cup m_{\mathrm{comp}}$ where $\mathrm{cs}(m'' * \mu) = \mathrm{cs}(\mu) + \Delta_{\mathrm{cs}}(m')$. The proof of reasonableness and non-contiguity of (S, m) is similar to the proofs in Proposition 126 with Lemma 117 replaced by Lemma 134.

To prove normality, suppose the first string s of m is continued above by S. Then $s \in m'$ and since m_{comp} cannot affect the rows affected by the strings of m', there must exist a string of η/μ that is the rightward shift of s. This implies that S contains the same number of boxes in each row r containing a box of s and also in the negatively modified row of s. Therefore, none of the rows containing a box of s is a modified row of S (given that the uppermost row containing a box of s is by hypothesis not a modified row of S), and also the negatively modified row of s is not a modified row of S.

If (S, m) is non-interfering then η/ν is a strip by Proposition 138.

If (S, m) is interfering then m_{comp} is a single string t. Suppose a cell x of t lies above a cell y of S. Since S is a horizontal strip y is λ-addable, and thus by hypothesis y is also a cell of m (t is a translate of the strings of m and it starts one row above the final string of m). Therefore η/ν is a horizontal strip. Obviously $\mathrm{cs}(\eta)/\mathrm{cs}(\nu) = \mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$ is a vertical strip so it only remains to show that $\mathrm{rs}(\eta)/\mathrm{rs}(\nu)$ is a horizontal strip. This is done as in the proof of Proposition 141. □

7. Pushout sequences

Consider an initial pair $(S, \mathbf{p})$ consisting of a strip μ/λ for $\lambda, \mu \in \Pi$ and a path $\mathbf{p}$ from λ to $\nu \in \Pi$. A *pushout sequence* from $(S, \mathbf{p})$ is a sequence of augmentation moves and pushouts which produces a final pair $(\tilde{S}, \mathbf{q})$ consisting of a *maximal* strip $\tilde{S} = \eta/\nu$ and a path $\mathbf{q}$ from μ to η for some $\eta \in \Pi$:

$$\begin{tikzcd} \lambda \arrow[r, "\mathbf{p}"] \arrow[d, "S"'] & \nu \arrow[d, "\tilde{S}"] \\ \mu \arrow[r, "\mathbf{q}"'] & \eta \end{tikzcd} \tag{70}$$

More precisely, a pushout sequence is defined by a diagram of the form

$$\begin{tikzcd} \lambda^0 \arrow[r, "m^1"] \arrow[d, "S^0"'] & \lambda^1 \arrow[r, "m^2"] \arrow[d, "S^1"'] & \cdots \arrow[r] & \lambda^{L-1} \arrow[r, "m^L"] \arrow[d, "S^{L-1}"'] & \lambda^L \arrow[d, "S^L"'] \\ \mu^0 \arrow[r, "n^1"'] & \mu^1 \arrow[r, "n^2"'] & \cdots \arrow[r] & \mu^{L-1} \arrow[r, "n^L"'] & \mu^L \end{tikzcd} \tag{71}$$

where $\lambda^0 = \lambda$, $S = S^0$, the top row of (71) consists of the path $\mathbf{p}$ (possibly with empty moves interspersed), each S^i is a strip with $\tilde{S} = S^L$ maximal, the n^i are (possibly empty) moves, the bottom row of (71) is the path $\mathbf{q}$, and for each $1 \le i \le L$, the diagram

$$\begin{tikzcd} \lambda^{i-1} \arrow[r, "m^i"] \arrow[d, "S^{i-1}"'] & \lambda^i \arrow[d, "S^i"] \\ \mu^{i-1} \arrow[r, "n^i"'] & \mu^i \end{tikzcd} \tag{72}$$

defines an augmentation move if m^i is empty, or the pushout of a compatible pair if m^i is not empty.

The main technical work in this paper is to establish the following existence and uniqueness properties of pushout sequences.

Proposition 145. *Each initial pair $(S, \mathbf{p})$ admits a canonical pushout sequence, which repeatedly maximizes the current strip and pushes out the resulting maximal strip with the next move, and ends with maximization.*

We prove Proposition 145 in Subsection 7.1 by giving an algorithm which computes the canonical pushout sequence.

Proposition 146. *Pushout sequences take equivalent paths to equivalent paths. That is, if $(S, \mathbf{p})$ and $(S, \mathbf{p}')$ are initial pairs with $\mathbf{p} \equiv \mathbf{p}'$ and there are pushout sequences from $(S, \mathbf{p})$ and $(S, \mathbf{p}')$ that produce the final pairs $(\tilde{S}, \mathbf{q})$ and $(\tilde{S}', \mathbf{q}')$ respectively, then $\tilde{S} = \tilde{S}'$ and $\mathbf{q} \equiv \mathbf{q}'$.*

It follows that pushout sequences define a map $(S, [\mathbf{p}]) \to (\tilde{S}, [\mathbf{q}])$ where $(S, \mathbf{p})$ is an initial pair and $(\tilde{S}, \mathbf{q})$ is a final pair with $\tilde{S}$ maximal, fitting the diagram (70).

The special case $\mathbf{p}' = \mathbf{p}$ of Proposition 146 is proved in Subsection 7.2. The general case is proved in Section 8.

7.1. **Canonical pushout sequence.** The following algorithm `PushoutSequence` produces a canonical pushout sequence from $(S = \mu/\lambda, \mathbf{p})$. It suffices to produce the path $\mathbf{q}$, as the output strip $\tilde{S}$ is defined by the last elements of $\mathbf{p}$ and $\mathbf{q}$. We may assume that $\mathbf{p} = (\lambda = \lambda^0, \lambda^1, \ldots, \lambda^L)$ has no empty moves and m^i is the move from λ^{i-1} to λ^i. Let

$$\texttt{PushoutCompatiblePair}(\rho, \lambda^{i-1}, \lambda^i)$$

compute the following pushout and return η

$$\begin{tikzcd} \lambda^{i-1} \arrow[r, "m^i"] \arrow[d] & \lambda^i \arrow[d, dotted] \\ \rho \arrow[r, dotted] & \eta \end{tikzcd} \tag{73}$$

as specified in Subsections 5.4 and 5.5 if m^i is a row move and 6.5 and 6.6 if m^i is a column move.

proc `PushoutSequence`(μ, λ, p):
  local $q := (\mu)$, q'
  $\rho := \mu$
  for i **from** 1 to **length**(p):
    $q' = \texttt{MaximizeStrip}(\rho, \lambda^{i-1})$
    extend q by q'
    $\rho :=$ **last**(q')
    $\rho := \texttt{PushoutCompatiblePair}(\rho, \lambda^{i-1}, \lambda^i)$
    append ρ to q
  $q' := \texttt{MaximizeStrip}(\rho, \lambda)$
  extend q by q'
  return q

This procedure builds up a path q, implemented as a list of shapes. The variable q is initialized to be the list with a single item μ. For each move m^i in p, the current strip is maximized. By Propositions 125 and 141, the resulting initial pair is compatible and hence its pushout with the current move is well-defined. The output strip (given by the last shapes in q and p respectively) is maximal due to the last invocation of `MaximizeStrip`. The "extension" step takes the path q, given as a list of k-shapes, and extends it by the path q'. Note that the last element of q equals the first element of q'.

7.2. **Pushout sequences from (S, p) are equivalent.** In this subsection we prove the following result, which is the $\mathbf{p} = \mathbf{p}'$ case of Proposition 146.

Proposition 147. *Let $S = \mu/\lambda$ be a strip and $\mathbf{p}$ a path in Π from λ to ν. Then any two pushout sequences from $(S, \mathbf{p})$ produce the same strip and equivalent paths.*

We shall reduce the proof of Proposition 147 to that of Proposition 148 and then use the rest of the subsection to prove the latter.

Consider the setup of Proposition 147. By induction on the number of moves in $\mathbf{p}$ we may assume that $\mathbf{p} = m$ is a single move. We may assume that one of the pushout sequences to be compared, is the canonical one, which first passes from the strip S to its maximization $S_{\max}$ by the augmentation path $\mathbf{r}$, then does the

pushout $\mathrm{push}(S_{\max}, m) = (S'', \tilde{m})$, and finally maximizes the resulting strip via the augmentation path $\tilde{\mathbf{r}}$, resulting in the maximal strip $\tilde{S}$ and the path $\mathbf{q} = \tilde{\mathbf{r}}\tilde{m}\mathbf{r}$.

Consider any other pushout sequence from (S, m), which produces $(\tilde{S}', \mathbf{q}')$, say. Suppose the first operation in this pushout sequence is an augmentation move m'. The move m' is the first in the output path $\mathbf{q}'$; let the path $\tilde{\mathbf{q}}$ be the rest of $\mathbf{q}'$. Let $\mathbf{t}$ be any augmentation path from $m' \cup S$ to its maximization. By Proposition 91, this maximization is equal to $S_{\max}$ and $\mathbf{t}m' \equiv \mathbf{r}$. We have

$$\mathbf{q}' = \tilde{\mathbf{q}}m' \equiv \tilde{\mathbf{r}}\tilde{m}\mathbf{t}m' \equiv \tilde{\mathbf{r}}\tilde{m}\mathbf{r} = \mathbf{q}, \tag{74}$$

which holds by induction since $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}\tilde{m}\mathbf{t}$ are equivalent, being produced from the same pair $(m' \cup S, m)$ by pushout sequences, with $m' \cup S$ closer to maximal than S.

We may therefore assume that the first operation in the pushout sequence producing $(\tilde{S}', \mathbf{q}')$, is a pushout, and in particular that (S, m) is compatible. Let $\mathrm{push}(S, m) = (S', M)$. Writing $\mathbf{q}' = \tilde{\mathbf{q}}M$, $\tilde{\mathbf{q}}$ is an augmentation path that maximizes S' and produces $\tilde{S}'$.

We may also assume that S is not already maximal, for otherwise there is only one way to begin the pushout sequence from (S, m). Then $\mathbf{r}$ is nonempty; let its first move be x and $\mathbf{r}'$ the remainder of $\mathbf{r}$. Since x is a move in the canonical maximization of S, it is a maximal completion move that augments S.

We apply Proposition 148, using the label $S' \cup \tilde{x}$ for the front right upward arrow. Let $\mathbf{y}$ be an augmentation path that maximizes the strip $S' \cup \tilde{x}$. We have

$$\mathbf{q}' = \tilde{\mathbf{q}}M \equiv \mathbf{y}\tilde{x}M \equiv \mathbf{y}\tilde{M}x \equiv \tilde{\mathbf{r}}\tilde{m}\mathbf{r}'x = \mathbf{q}.$$

The first equivalence holds by Proposition 91 since both $\tilde{\mathbf{q}}$ and $\mathbf{y}\tilde{x}$ are maximizations of S'. The second holds by the equivalence of the top face of (75) in Proposition 148. The third equivalence holds by induction since $\mathbf{y}\tilde{M}$ and $\tilde{\mathbf{r}}\tilde{m}\mathbf{r}'$ are equivalent, being the paths produced by two pushout sequences from $(S \cup x, m)$ with $S \cup x$ closer to maximal than S.

Thus we have reduced the proof of Proposition 147 to that of Proposition 148.

Proposition 148. *Let $(S = \mu/\lambda, m = \nu/\lambda)$ be a compatible initial pair with $\mathrm{push}(S, m) = (S', M)$ and let $x = \kappa/\mu$ be a maximal completion move that augments S. Then we have the commuting cube*

$$\begin{tikzcd}
 & \mu \arrow[rr, "x"] \arrow[dl, "M"'] & & \kappa \arrow[dl, dotted, "\tilde{M}"'] \\
\cdot \arrow[rr, dotted, "\tilde{x}"] & & \eta & \\
 & \lambda \arrow[uu, "S"'] \arrow[rr, "\emptyset"] \arrow[dl, "m"] & & \lambda \arrow[uu, "S\cup x"'] \arrow[dl, "m"] \\
\nu \arrow[uu, "S'"] \arrow[rr, "\emptyset"'] & & \nu \arrow[uu, dotted, "\tilde{S}"'] &
\end{tikzcd} \tag{75}$$

in which vertical edges are strips and other edges are moves, the left and right faces are pushouts, the front and back faces are augmentations, and the top face is an elementary equivalence.

Lemma 149. *Let x be a maximal completion row move and m a row move on λ such that x and m interfere and x is above m. Then (x, m) is lower-perfectible.*

Proof. Let $a = (r, c)$ be the lowest cell of the initial string t of x. Then c^- is a negatively modified column of m and $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c^-} - 1$. We claim that $b = \mathrm{bot}_c(\partial\lambda)$ and $b^- = \mathrm{bot}_{c^-}(\partial\lambda)$ are on the same row. This follows from the estimate $h_\lambda(b) \geq h_\lambda(b^-) - 1 = k - 1$ and the assumption that t is maximal. It follows that there is a $(m * \lambda)$-addable corner in the row containing b^- and b by Remark 13. The rest of the proof is similar to Lemma 93. □

Lemma 150. *Let x be a maximal completion column move and m a column move from λ such that x and m interfere and x is below m. Then (x, m) is the transpose analogue of a lower-perfectible interfering pair of row moves.*

Proof. The proof is similar to that of Lemma 149. □

Lemma 151. *Let $(S = \mu/\lambda, m = \nu/\lambda)$ be a compatible initial pair,* $\mathrm{push}(S, m) =$ (S', M) *and let $x = \eta/\mu$ be a maximal completion move for S. Then x and M define an elementary equivalence.*

Proof. By the definition of pushout, if m is a row (resp. column) move then M is either a row (resp. column) move or empty. In some cases it will be shown that $(S \cup x, m)$ is compatible. In that case we write $\mathrm{push}(S \cup x, m) = (\tilde{S}, \tilde{M})$.

I) m is a row move and x a maximal completion row move.

Suppose first that x and M intersect. Then the first string $s \in x$ must intersect a string t of M. Since x is maximal, M cannot continue below x. Let us suppose that M continues above x. Let b' be the first cell in s; it is a lower augmentable corner for some modified column c of S containing a cell a. It follows that t must contain a cell b in column c (on top of a). But $b \in S'$ as well and S' is a horizontal strip so $a \in m$. It is clear that t must be part of m_{comp}, but this contradicts Lemma 127. Thus if x and M intersect, they satisfy an elementary row equivalence.

Now suppose that x and M interfere. If x is above M then by Lemma 149 a lower perfection M_{per} exists. Note that we can then easily check that $(S \cup x, m)$ is compatible. Then $\tilde{M} = M \cup M_{\mathrm{per}}$ and $\tilde{S} = S' \cup x \cup M_{\mathrm{per}}$ (see (75)). If M occurs above x, let c be the leftmost positively modified column of M. By definition of pushout, c is not a positively modified column of S. And for x to be a completion move, c^- needs to be a positively modified column of S. Since $\mathrm{cs}(\mu)_c = \mathrm{cs}(\mu)_{c^-} - 1$ we thus have $\mathrm{cs}(\lambda)_c = \mathrm{cs}(\lambda)_{c^-}$. Therefore for c to be a positively modified column of M, all the negatively modified columns of x had to be positively modified columns of m. Therefore, the strings of m that are contiguous to strings of x are all continued above and below in $S \cup x$. Since (S, m) is compatible, this gives that $(S \cup x, m)$ is compatible with $\tilde{M} = M \cup M_{\mathrm{per}}$ and $\tilde{S} = S' \cup x \cup M_{\mathrm{per}}$, where M_{per} is given by the strings of m that are contiguous to strings of x pushed above one cell. It is then easy to see that (x, M) is upper perfectible by M_{per}.

If M is empty then it is easy to check that $(S \cup x, m)$ is compatible (with pushout $(\tilde{S}, \tilde{M})$, say) such that either $\tilde{M}$ is empty or $\tilde{M}$ is a row move from $x * \mu$ and $\tilde{x} := \tilde{M} \cup x$ is a row move from μ with $\tilde{M}$ extending the strings of x above. Either way we obtain an elementary equivalence $\tilde{x}M \equiv \tilde{M}x$.

II) m is a row move and x is a maximal completion column move.

Let M and x be intersecting moves. We show that x continues above and below M, so that M and x satisfy an elementary equivalence. Since a row and a column move cannot be matched above and cannot be matched below, it suffices to show

that M does not continue above x and does not continue below x. Suppose that M continues above x. Let b be the highest cell of $M \cap x$ and s the string in M containing b. Let a be the cell above and contiguous to b in s. By the maximality of x, a has to lie above a cell of S. So the string s of M was pushed above during the pushout. Hence the cell below b is also in S. This contradicts the fact that x is a completion move. Suppose that M continues below x. Let b be the lowest cell of x and let s be the string of M containing b. The cell b is an upper augmentable corner for some modified row R of S. Since M continues below x, the string s contains a cell b' in row R. The string s of M cannot have come from pushing above a string of m since the cell below b is not in S by definition of upper augmentable corner. If $s \subset m$ then all cells of S in row R are also in m and thus row R cannot be a modified row of S by reasonableness. Thus $s \subset m_{\mathrm{comp}}$. Since the cell to the left of b' lies in μ, it cannot also lie in m_{comp}. So all the cells outside of λ and to the left of b' lie in $m \cap S$. But (S, m) is reasonable, so row R would not be a positively modified row of S, a contradiction.

Suppose M and x do not intersect and are contiguous. In this case M is above x. Let b be the highest cell of x and a the cell of M contiguous with b. By maximality of x, a has to lie above a cell a' of S. But since this implies that the string s of M that contains a was pushed above during the pushout, we have that the column of a' is not a modified column of S. Therefore there needs to be a cell of S below b. But this is a contradiction to the fact that x is a completion move.

Suppose M is empty. In this case one may deduce that $(S \cup x, m)$ is compatible with $\tilde{M}$ empty. Then x and M satisfy a trivial equivalence.

III) m is a column move and x is a maximal completion row move.

Let M and x be intersecting. By maximality of x, x continues below M. Suppose that M continues above x. Let b be the highest cell of x. It is a lower augmentable corner of S associated to a modified column c of S. Since M continues above, there is a cell a of M in column c that lies above a cell of S. Suppose a belongs to m^+. By Lemma 137, a does not belong to the first string of m^+ and so there is a cell of m^+ in the row below that of a. Given that S is a horizontal strip, a lies above a cell of $m \cap S$ and so does b by reasonableness and translation of strings in a move. But then we have the contradiction that b lies above a cell of S. Therefore $a \in m_{\mathrm{comp}}$. By Lemma 142 the cells of m_{comp} are in the same column as the final string of m and thus we get again the contradiction that b lies above a cell of $m \cap S$. Therefore if M and x intersect x continues above and below M.

Suppose M and x do not intersect and M and x are contiguous. M cannot be below x due to the maximality of x. If M is above x then a contradiction is reached as in the previous paragraph.

Suppose M is empty. Then m is a single string that is matched above by S and (S, m) is non-interfering. Since x is a completion row move for S, it follows that m is matched above by $S \cup x$ and $(S \cup x, m)$ is non-interfering. Therefore $(S \cup x, m)$ is compatible with $\tilde{M}$ empty.

Hence M and x satisfy an elementary equivalence.

IV) m is a column move and x a maximal completion column move.

Let M and x be intersecting. Suppose M continues above x. Let b be the highest cell of x and let a be the cell of M contiguous to it from above. By maximality of x, cell a lies above a cell of S. Therefore by Lemma 137 a belongs to m_{comp} and we have as before the contradiction that x lies above a cell of m. Suppose M

continues below x. Let s be the string of M that meets x. The bottommost cell a of x is an upper augmentable corner of S associated to row R, say. The cell a is also in s and since M continues below x, s has a cell b contiguous to and below a. Since row R has a μ-addable cell contiguous to a, we deduce that it is b. The cell b cannot belong to m since m is a vertical strip and there are cells of S to the left of b since R is a modified row of S. Neither can b belong to m^+ since in this case R could not be a modified row of S by normality. So a and b belong to m_{comp}. Since m_{comp} does not lie above the last string of m (otherwise there would be a cell of S below a), there is an upper augmentable corner of S below a by Lemma 143. This contradicts the assumption that x is a maximal completion column move.

Now suppose that x and M interfere. As in (I), Lemma 150 covers the case where M is above x. And if M is below x, the transpose of the argument given in (I), shows that (M, x) satisfies the transpose analogue of an upper-perfectible pair of interfering row moves.

Suppose M is empty. Then m consists of a single string that is matched above by S and (S, m) is non-interfering. Since m is also contained in $S \cup x$ we see that $(S \cup x, m)$ is normal. Suppose m is matched above by $S \cup x$. The case that special interference occurs for $(S \cup x, m)$, is handled by Lemma 152 below; in particular M and x satisfy an elementary equivalence. Otherwise $(S \cup x, m)$ is non-interfering and therefore compatible. Then $\tilde{M}$ is empty, which leads to a trivial elementary equivalence for M and x. Otherwise m is continued above by $S \cup x$. Then the negatively modified row of x is the positively modified row of m (say the r-th) and $\Delta_{\mathrm{rs}}(S)_r = 1$. In this case one may deduce the noninterference of $(S \cup x, m)$ from that of (S, m). Therefore $(S \cup x, m)$ is compatible. We have $\tilde{M} = m^+$ (where m^+ is defined for the pair $(S \cup x, m)$) which is continued above by x to be a column move $x \cup \tilde{M}$ from μ.

Hence M and x satisfy an elementary equivalence. □

Lemma 152. *Suppose $m = s$ is a column move such that $\mathrm{push}(S, m) = (S \setminus m, \emptyset)$, and suppose that x is a maximal completion column move from μ such that $(S \cup x, m)$ is in the special interference case. Then $(S \cup x, m)$ is pushout-perfectible, with m_{comp} such that $\mathrm{push}(S \cup x, m) = ((S \cup x \cup m_{\mathrm{comp}}) \setminus m, m_{\mathrm{comp}})$. Moreover m_{comp} corresponds to m shifted up one cell and m_{comp} extends x above to a column move from μ.*

Proof. Let $\eta = x * \mu$. By assumption $m \subset S$, the single string of m matches $S \cup x$ above, and $\mathrm{rs}(\eta)/\mathrm{rs}(\nu)$ is not a horizontal strip where $\nu = m * \lambda$.

In this case, we must have $\mathrm{rs}(\eta)_{R^+} = \mathrm{rs}(\mu)_R$, with R the negatively modified row of m and R^+ the positively modified row of x. Let b be the leftmost cell in $\partial\eta$ in row R^+, and let c be the leftmost cell in $\partial\lambda$ in row R. We claim that b and c lie in the same column.

Since the hook-length of c in λ is k by definition of moves, we have from $\mathrm{rs}(\mu)_R = \mathrm{rs}(\eta)_{R^+}$ that the hook-length of b in η is $k-1$ or k. In the case that it is equal to $k-1$ we easily see that b and c lie in the same column. In the other case, the claim follows from Lemma 101(3).

The rest of the proof is then exactly as in the proof of Proposition 141. □

Proof of Proposition 148. The existence of an equivalence $\tilde{M}x = \tilde{x}M$ is guaranteed by Lemma 151. In some cases (when $M = \emptyset$ or when (x, M) interferes) there may be more than one choice for such an equivalence. In such cases, the proof of Lemma

151 provides a particular $\tilde{M}$. In the other cases M and x define a unique elementary equivalence which uniquely specifies $\tilde{M}$.

It suffices to show that $(S\cup x, m)$ is compatible and that $\text{push}(S\cup x, m) = (\tilde{S}, \tilde{M})$ for some strip $\tilde{S}$ and with $\tilde{M}$ specified as above. It will then follow that $\tilde{x}$ augments S' to give $\tilde{S}$.

Let κ and η be defined by $S \cup x = \kappa/\lambda$ and $\eta = \tilde{M} * \kappa$. We will use the criteria of Propositions 126 and Proposition 144. It is clear that (when it is nonempty) $\tilde{M}$ has the same diagram as M which has the same diagram as m. It is also clear from the commutativity of the top face that $\nu \subset \eta$. It remains to check Condition (1) of Proposition 126 (or Proposition 144).

I) m and x are row moves. The proof of Lemma 151 deals with the cases where (x, M) is interfering. If $(S \cup x, m)$ interferes while (S, m) does not, then x and M interfere and this case has already been covered. Suppose that (S, m) interferes while $(S \cup x, m)$ does not. In this case the negatively modified columns of m are immediately to the left of the negatively modified columns of x, and we have that x and M are matched above with x continuing below M. So we get

$$\begin{aligned}
\text{ecs}(S \cup x, m) &= \text{cs}(\lambda) + \Delta_{\text{cs}}(S \cup x) + \Delta_{\text{cs}}(m') \\
&= \text{cs}(\mu) + \Delta_{\text{cs}}(x) + \Delta_{\text{cs}}(m') \\
&= \text{cs}(\mu) + \Delta_{\text{cs}}(\tilde{x}) + \Delta_{\text{cs}}(M).
\end{aligned}$$

If some (but not all) positively modified columns of m are negatively modified columns of x, then M and x interfere. Hence this case has already been covered. If the positively modified columns of m are all negatively modified columns of x, then $M = \emptyset$, which was covered in the proof of Lemma 151.

Finally, in all the other cases (x, M) does not interfere and M is nonempty, so that there is a unique choice for the equivalence $\tilde{x}M \equiv \tilde{M}x$. Moreover, the positively modified columns of m are not negatively modified columns of x, and if there is interference in (S, m) and $(S\cup x, m)$ then it will occur in the same positions and require the same perfection (that is, the interference has nothing to do with x). Let m' be defined as usual when calculating the pushout $(S\cup x, m)$. Let m_1 be the strings of m matched below by S. Let m_2 be the strings of m matched below by x. Then $\Delta_{\text{cs}}(m') = \Delta_{\text{cs}}(m) - \Delta_{\text{cs}}(m_1 \cup m_2)$ and we calculate:

$$\begin{aligned}
\text{ecs}(S \cup x, m) &= \text{per}_m(\text{cs}(\lambda) + \Delta_{\text{cs}}(S \cup x) + \Delta_{\text{cs}}(m')) \\
&= \text{per}_m\big(\text{cs}(\lambda) + \Delta_{\text{cs}}(S) + \Delta_{\text{cs}}(x) + \Delta_{\text{cs}}(m')\big) \\
&= \text{per}_m\big(\text{cs}(\mu) + \Delta_{\text{cs}}(x) + \Delta_{\text{cs}}(m) - \Delta_{\text{cs}}(m_1 \cup m_2)\big) \\
&= \text{per}_m\big(\text{cs}(\mu) + \Delta_{\text{cs}}(m) - \Delta_{\text{cs}}(m_1) + \Delta_{\text{cs}}(x) - \Delta_{\text{cs}}(m_2)\big) \\
&= \text{per}_m\big(\text{cs}(\mu) + \Delta_{\text{cs}}(m) - \Delta_{\text{cs}}(m_1)\big) + \Delta_{\text{cs}}(\tilde{x}) \\
&= \text{cs}(\mu) + \Delta_{\text{cs}}(M) + \Delta_{\text{cs}}(\tilde{x}).
\end{aligned}$$

II) m is a row move and x is a column move. Notice that $\Delta_{\text{cs}}(S) = \Delta_{\text{cs}}(S \cup x)$ and the strips S and $S \cup x$ modify the same columns. Therefore $\text{ecs}(S \cup x, m) = \text{ecs}(S, m) = \text{cs}(\eta)$ and the result follows immediately.

III) m is a column move and x is a row move. This is similar to case *II)*.

IV) m and x are column moves. This case is basically the same as case *I)*, except that special care needs to be taken when there is special interference. Suppose there is special interference in $(S\cup x, m)$ but none in (S, m). In this case, if $m \subset S$, then

we are done by Lemma 152. If $m \cap S = \emptyset$, then $m \subseteq x$ and (S, m) were interfering (not special interference). Therefore $\mathrm{push}(S, m) = (S \cup m_{\mathrm{comp}}, m \cup m_{\mathrm{comp}})$ giving $\mathrm{push}(S \cup x, m) = ((S \cup x \cup m_{\mathrm{comp}}) \setminus m, m_{\mathrm{comp}})$ and the equivalence $m_{\mathrm{comp}}x = (x \setminus m)(m \cup m_{\mathrm{comp}})$ completing the cube.

Suppose there is special interference in (S, m) but none in $(S \cup x, m)$. In this case the negatively modified row of m is immediately below the negatively modified row of x, and we have that x and M are matched below. So we have as in case *I)*

$$\begin{aligned}
\mathrm{ers}(S \cup x, m) &= \mathrm{rs}(\lambda) + \Delta_{\mathrm{rs}}(S \cup x) + \Delta_{\mathrm{rs}}(m') \\
&= \mathrm{rs}(\mu) + \Delta_{\mathrm{rs}}(x) + \Delta_{\mathrm{rs}}(m') \\
&= \mathrm{rs}(\mu) + \Delta_{\mathrm{rs}}(\tilde{x}) + \Delta_{\mathrm{rs}}(M).
\end{aligned}$$

All the other cases are as in case *I)*. □

8. Pushouts of equivalent paths are equivalent

The goal of this section is to prove Proposition 146. By Proposition 147 it suffices to show that *there exist* pushout sequences starting from $(S, \mathbf{p})$ and $(S, \mathbf{p}')$ respectively, such that the resulting final pairs $(\tilde{S}, \mathbf{q})$ and $(\tilde{S}', \mathbf{q}')$, satisfy $\tilde{S} = \tilde{S}'$ and $\mathbf{q} \equiv \mathbf{q}'$. By Proposition 145 we may assume that both pushout sequences start by maximizing the strip S. We may therefore assume that S is already maximal. Since equivalences in the k-shape poset are generated by elementary equivalences, we may assume by induction on the length of the paths, that $\mathbf{p} = \tilde{n}m \equiv \tilde{m}n = \mathbf{p}'$ is an elementary equivalence starting at λ.

To summarize, it suffices to show that given the elementary equivalence $\tilde{n}m \equiv \tilde{m}n$ starting at λ and a λ-addable maximal strip S, there exist pushout sequences from $(S, \tilde{n}m)$ and $(S, \tilde{m}n)$, producing $(\tilde{S}, \tilde{N}M)$ and $(\tilde{S}', \tilde{M}N)$ respectively, such that $\tilde{S} = \tilde{S}'$ and $\tilde{N}M \equiv \tilde{M}N$.

Since S is maximal, (S, m) and (S, n) are compatible. Let $\mathrm{push}(S, m) = (S_m, M)$ and $\mathrm{push}(S, n) = (S_n, N)$. This furnishes the three faces touching the vertex λ in the cube pictured in (76), and all vertices except ω.

$$\begin{tikzcd}
& \mu \arrow[rr, "N"] \arrow[dl, "M"'] & & \rho \arrow[dl, dotted, "\tilde{M}"'] \\
\eta \arrow[rr, dotted, "\tilde{N}"] & & \omega & \\
& \lambda \arrow[uu, "S"] \arrow[rr, "n"'] \arrow[dl, "m"] & & \kappa \arrow[uu, "S_n"'] \arrow[dl, "\tilde{m}"] \\
\nu \arrow[uu, "S_m"] \arrow[rr, "\tilde{n}"'] & & \theta \arrow[uu, dotted, "\tilde{S}"'] &
\end{tikzcd} \tag{76}$$

It suffices to prove the following.

(1) If $\tilde{n} \neq \emptyset$ then $(S_m, \tilde{n})$ is compatible. Let $\mathrm{push}(S_m, \tilde{n}) = (\tilde{S}, \tilde{N})$ with final shape ω.
(2) If $\tilde{m} \neq \emptyset$ then $(S_n, \tilde{m})$ is compatible. Let $\mathrm{push}(S_n, \tilde{m}) = (\tilde{S}', \tilde{M})$ with final shape ω'.
(3) We may assume not both $\tilde{m}$ and $\tilde{n}$ are empty.
    (a) If $\tilde{n} \neq \emptyset$ and $\tilde{m} \neq \emptyset$ then $\omega = \omega'$ and $\tilde{N}M \equiv \tilde{M}N$ is an elementary equivalence.

(b) If $\tilde{n} \neq \emptyset$ and $\tilde{m} = \emptyset$ then (with ω defined by (1)) ω/ρ is a move and the right face of (76) defines an augmentation move.
(c) If $\tilde{m} \neq \emptyset$ and $\tilde{n} = \emptyset$ then (with $\omega = \omega'$ defined by (2)) ω/η is a move and the front face of (76) defines an augmentation move.

8.1. **Pushout of equivalences.**

Lemma 153. *Suppose (m, n) is an interfering pair of row (resp. column) moves from λ which is (either lower- or upper-) perfectible by adding the set of cells mn_{per}. Suppose $S = \mu/\lambda$ is a maximal strip. Then for each string $s \in mn_{\text{per}}$, we have $s \cap S = \emptyset$ or $s \subset S$.*

Proof. Let s be a string of mn_{per} and let $x, y \in s$ be contiguous cells with $x \in S$ while $y \notin S$. Suppose x is above y. We may assume that y is μ-addable, for otherwise we may shift left or down to another string of mn_{per}. Then the column of x is a modified column of S and so y is a lower augmentable corner of S, a contradiction. Suppose x is below y. Then the row of x is a modified row of S and so y is an upper augmentable corner of S, again a contradiction. □

Lemma 154. *Suppose M and N interfere. Then so do m and n.*

Proof. We first suppose that M and N are row moves and we assume without loss of generality that M is above N. Let c_N be the rightmost negatively modified column of N, so that c_N^+ is the leftmost positively modified column of M. Since M and N interfere we have $\text{cs}(\mu)_{c_N} = \text{cs}(\mu)_{c_N^+} + 1$. Now let c_n be the rightmost negatively modified column of n, and let c_m be the leftmost positively modified column of M. Since strings of M and N are translates of those of m and n we have $\text{cs}(\mu)_{c_N} = \text{cs}(\lambda)_{c_n}$ and $\text{cs}(\mu)_{c_N^+} = \text{cs}(\lambda)_{c_m}$. But then $\text{cs}(\lambda)_{c_n} = \text{cs}(\lambda)_{c_m} + 1$ so m and n interfere.

When M and N are column moves, the proof is similar. □

Lemma 155. *Let (m, n) be a pair of moves from λ that define an elementary equivalence and let $S = \mu/\lambda$ be a maximal strip. Write* $\text{push}(S, m) = (S_m, M)$ *and* $\text{push}(S, n) = (S_n, N)$. *Then the pair (M, N) defines an elementary equivalence.*

Proof. I) m and n are row moves. We may assume that M and N are nonempty. By Lemma 112 the final string of m and of n is not matched below by S.

Suppose M and N do not intersect. Since both are row moves they are not contiguous. If (M, n) is non-interfering then they satisfy an elementary equivalence. So we may assume that (M, N) is interfering. By Lemma 154, (m, n) is interfering. Without loss of generality let m be above n. First suppose (m, n) is lower-perfectible by adding the set of cells mn_{per}. If (S, m) is non-interfering then by Lemma 153, (M, N) is lower-perfectible where MN_{per} lies in a subset of the columns of mn_{per} (even if (S, n) interfered). If (S, m) is interfering then (M, N) is lower-perfectible; the required additional strings for MN_{per} (which lie on the same set of rows; see Proposition 125) can be constructed using the technique of Lemma 93. If (m, n) is upper-perfectible, one may similarly show that (M, N) is upper-perfectible.

Otherwise we may assume that M and N intersect. By the definition of row equivalence we may assume that there exist strings s and t of M and N respectively such that s continues above t while t continues below s.

Suppose the string s (resp. t) belongs to m_{comp} (resp. n_{comp}) and the string t (resp. s) comes from a string of n (resp. of m) that was pushed above. Then we obtain the contradiction that S_m (resp. S_n) is not a horizontal strip.

Suppose the string s (resp. t) belongs to m_{comp} (resp. n_{comp}) and the string t (resp. s) is a string of n (resp. m). By Proposition 125 m_{comp} (resp. n_{comp}) lies on the rows of the last string of m (resp. n), yielding the contradiction that m and n already intersected and did not satisfy an elementary equivalence.

Suppose the string s (resp. t) belongs to m (resp. n) and the string t (resp. s) was pushed above. This is a contradiction since m, n, and S are all λ-addable.

In all other cases, one deduces the contradiction that m and n meet but do not satisfy an elementary equivalence.

II) m is a row move and n is a column move. It suffices to check that M and N are reasonable and non-contiguous.

Suppose M and N are not reasonable. Let s and t be strings of M and N respectively that are not reasonable.

Suppose the strings s and t belong to m_{comp} and n_{comp} respectively. Let $a \in s \cap t$. By Proposition 141, $a \in s = n_{\mathrm{comp}}$ lies atop a cell b of n. In particular $b \notin \lambda$. Since $a \in M$ and M is a $\lambda \cup S$-addable horizontal strip, $b \in \lambda \cup S$, that is, $b \in S$. Then we have the contradiction that either S_m is not a horizontal strip or m and n already intersected and did not satisfy an elementary equivalence.

Suppose the strings s and t come from strings of m and n that have been pushed up and right respectively. Then we have the contradiction that either S_n is not a horizontal strip or m and n already intersected and did not satisfy an elementary equivalence.

Suppose the string s belongs to m_{comp} and the string t is a string of n. By Proposition 125 m_{comp} lies on the rows of the last string of m. This leads to the contradiction that m and n already intersected and did not satisfy an elementary equivalence.

All the other cases can easily be ruled out.

Now suppose M and N are contiguous. Suppose M is above N. Let x and y be cells of M and N respectively that are contiguous. Suppose $y \in n_{\mathrm{comp}}$. By Proposition 141 it follows that the cell y^- just below y is in n and $\mathrm{row}(y^-)$ is a modified row of n. This implies that the cell below x does not belong to λ and thus needs to belong to S. Therefore x cannot be part of m_{comp} since otherwise S_m would not be a horizontal strip by Lemma 127. So the string that contains x was pushed up during the pushout. But then we have the contradiction that m and n are contiguous.

Suppose that y belongs to a string of n that was pushed right during the pushout. In this case the row of y is not a modified row of S and we have that the cell x^- immediately to the left of x is also in S, but $x^- \notin m$ for otherwise m and n would already be contiguous. By Proposition 125 it follows that $x \notin m_{\mathrm{comp}}$. Since $x^- \in S$ it follows that x was pushed up during $\mathrm{push}(S, m)$. $c = \mathrm{col}(x)$ is not a modified column of S and we get that the cell below y is also in S, which yields the contradiction $\mathrm{cs}(\lambda)_{c^-} = \mathrm{cs}(\lambda)_c$.

Finally, suppose that y belongs to n. Since m and n are not contiguous x does not belong to m. If $x \in m_{\mathrm{comp}}$ then by Proposition 125 there is a cell z in its row that belongs to m. But then a hook-length analysis shows that the column of z cannot be a modified column of m, a contradiction. So x belongs to a string of M

that was pushed up during the pushout. Hence the column of x is not a modified column of S and there is a cell of $S \cap n$ below y that is contiguous with the cell below x, contradicting the assumption that m and n are not contiguous.

The case in which M is below N, is similar.

III) m and n are column moves. The proof is similar to that of *I)* (using the fact that the perfection of a column move lies on the same columns as its final string). □

8.2. **Commuting cube (non-degenerate case).** Suppose that $m, n, \tilde{m}, \tilde{n}, M$ and N are non-empty. Then the following cube commutes

$$\begin{tikzcd}
 & \mu \arrow[rr, "N"] \arrow[dl, "M"'] & & \rho \arrow[dl, "\tilde{M}"'] \\
\eta \arrow[rr, "\tilde{N}"] & & \omega & \\
 & \lambda \arrow[uu, "S"] \arrow[rr, "n"] \arrow[dl, "m"] & & \kappa \arrow[uu, "S_n"'] \arrow[dl, "\tilde{m}"] \\
\nu \arrow[uu, "S_m"] \arrow[rr, "\tilde{n}"'] & & \theta \arrow[uu, "\tilde{S}"] &
\end{tikzcd} \tag{77}$$

so that the two horizontal faces are elementary equivalences and the four vertical faces are pushouts. The three faces touching λ are assumed to be given.

By Lemma 155, the top face defines an elementary equivalence. Since M and N are non-empty ω is determined uniquely.

We will use Proposition 126 (or Proposition 144) to show that there exists a $\tilde{S}$ such that $\mathrm{push}(S_m, \tilde{n}) = (\tilde{S}, \tilde{N})$ and $\mathrm{push}(S_n, \tilde{m}) = (\tilde{S}, \tilde{M})$.

Preliminary claim: *Conditions (2) and (3) of Proposition 126 (or Proposition 144) hold.*

It is obvious that conditions (2) holds since by definition of pushouts and equivalences, M and N are moves whose strings have the same diagrams respectively as m and n and thus $\tilde{M}$ and $\tilde{N}$ are moves whose strings have the same diagram respectively as $\tilde{m}$ and $\tilde{n}$ (M and N interfere in this case if and only if m and n interfere). We will now see that condition (3) also holds, that is that $\theta \subset \omega$. Suppose m and n are row moves. Strings of $\tilde{m}$ that are strings of m are obviously contained in ω. Suppose s^+ is a string of $\tilde{m}$ that corresponds to a string s of m that has been pushed up (let's say it intersected with string t of n) . Then $s \subset t \in n$ and we either have $t \subset S$ or $t \cap S = \emptyset$. In the former case, $s^+ \in M$ and thus $s^+ \subset \omega$. In the latter case, $s \in M \cap N$ with $t \in N$ and thus $s^+ \in \tilde{M}$, which gives $s^+ \subset \omega$. Finally, suppose that s is a string in the perfection mn_{per} of m and n. By Lemma 153, we either have $s \subset S$ or $s \cap S = \emptyset$. In the former case, obviously $s \subset \omega$. In the latter case, since M and N are not empty by hypothesis, we have that (M, N) interferes. In every case s will belong to MN_{per} and will thus belong to ω. If m and n are column moves, or if m is a row move and n is a column move, then condition (3) is shown in a similar way.

To check that the vertical faces are pushouts, it remains to verify condition (1) of Proposition 126 (or Proposition 144).

We will use the fact (see the proof of Lemma 155) that if (m, n) is interfering and lower (resp. upper) perfectible and (M, N) is interfering, then (M, N) is lower (resp. upper) perfectible.

I) m **and** n **are row moves.** Let m_{comp} and n_{comp} be the sets of cells added in the pushout perfections of (S, m) and (S, n) respectively; they are empty in the non-interfering case. Also let mn_{per} denote the set of cells defining the lower or upper perfection of (m, n), if it exists. We will repeatedly use the fact (Proposition 125) that m_{comp} (resp. n_{comp}) all lie on the same row as the last string of m (resp. n).

Main claim:

$$\mathrm{ecs}(S_m, \tilde{n}) = \mathrm{cs}(\omega) = \mathrm{ecs}(S_n, \tilde{m}) \tag{78}$$

To prove (78), it suffices to make a calculation with modified columns.

We shall be dividing our study into four cases according to the type of row equivalence: m and n do not interact, m and n are matched below, m and n are matched above, and m and n interfere.

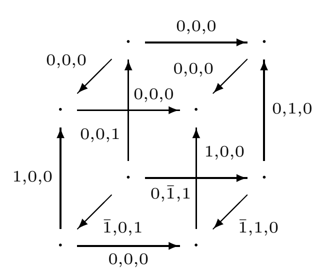
8.2.1. *m and n do not interact.* By Lemma 154, M and N do not interfere. Furthermore, since m_{comp} (resp. n_{comp}) all lie on the same row as the last string of m (resp. n), we have that m_{comp} does not interact with n, n_{comp} does not interact with m, and m_{comp} does not interact with n_{comp}. In particular, M and N do not interact. It is thus not difficult to see that we obtain

$$\begin{aligned}\mathrm{ecs}(S_m, \tilde{n}) = \mathrm{cs}(\omega) = \mathrm{ecs}(S_n, \tilde{m}) = \mathrm{cs}(\lambda) &+ \Delta_{\mathrm{cs}}(S) + \Delta_{\mathrm{cs}}(m') + \Delta_{\mathrm{cs}}(n') \\ &+ \Delta_{\mathrm{cs}}(m_{\mathrm{comp}}) + \Delta_{\mathrm{cs}}(n_{\mathrm{comp}})\end{aligned}$$

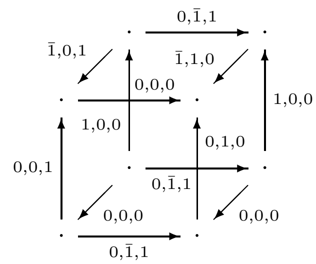
where m' and n' originate respectively from (S, m) and (S, n).

8.2.2. *m and n are matched below, with m continuing above n.* By Lemma 66, n has rank greater than m. By maximality of S, we see that m_{comp} and n_{comp} cannot intersect. Thus, by Lemma 66 applied to M and N, positively modified columns of m_{comp} are positively modified columns of n. We conclude that there are three interesting types of modified columns of S (the other types interact in a manner that was covered in 8.2.1): (a) those which are positively modified columns of both m and n, (b) those which are immediately to the right of negatively modified columns of m and cause interference, and (c) those which are immediately to the right of negatively modified columns of n and cause interference. Each such column will affect the vectors in (78) at three different indices: $(+)$ a positively modified column of $m \cup m_{\mathrm{comp}}$ or $n \cup n_{\mathrm{comp}}$, $(-m)$ a negatively modified column of $m \cup m_{\mathrm{comp}}$, or $(-n)$ a negatively modified column of $n \cup n_{\mathrm{comp}}$. For each of the cases (a), (b) and (c), we draw a cube whose edges give the entries $(-m), (-n), (+)$ associated to the corresponding move or strip in the cube (77). The commutation of the three cubes implies that (78) is satisfied. We will write $\bar{1}$ to denote a negatively modified column.

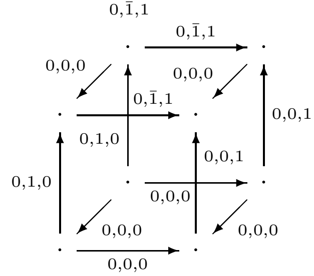
case (a)



case (b)



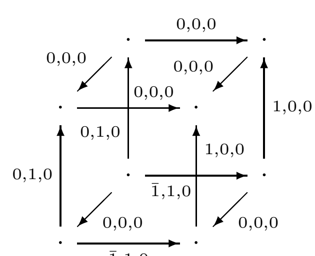
case (c)



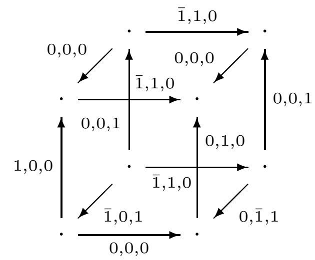
As an example of how to read these tables: if we look at a modified column c of S of type (a), there is a string $s \in m$ and a string $t \in n$ which end on the same column. Suppose the negatively modified column of s is c' and that of t is c''. Then reading the edge corresponding to $\tilde{m}$ in the cube in case (a), we get that the changes in cs across $\tilde{m}$ in columns c', c'', and c, are -1, 1, and 0 respectively.

8.2.3. *m and n are matched above, with m continuing below n.* By Lemma 66, n has greater rank than m. The rightmost positively modified column of n agrees with that of the rightmost negatively modified column of $\tilde{m}$. Therefore, if $\mathrm{push}(S_n, \tilde{m})$ involves interference due to cells of $S \cap S_n$, then one can deduce that all positively modified columns of n are positively modified columns of S. But this implies that $n' = \emptyset$ in the pushout of (S, n), which (without the existence of a modified column of S which causes interference with m and n) leads to an empty move N which we assume is not the case. We conclude that there are three interesting types of modified columns of S: (a) those which are positively modified columns of n but not m, (b) those which are positively modified columns of m but not n, and (c) those which are immediately to the right of negatively modified columns of both m and n and cause interference (with both m and n). Note that case (b) is especially interesting: interference always occurs when calculating $\mathrm{push}(S_m, \tilde{n})$. We list vectors in the indices: $(-)$ negatively modified columns of $n \cup n_{\mathrm{comp}}$, $(+n)$ positively modified columns of $n \cup n_{\mathrm{comp}}$, and $(+m)$ positively modified columns of $m \cup m_{\mathrm{comp}}$. For each of the cases (a), (b) and (c), we draw a cube whose edges give the entries $(-), (+n), (+m)$ associated to the corresponding move or strip in the cube (77).

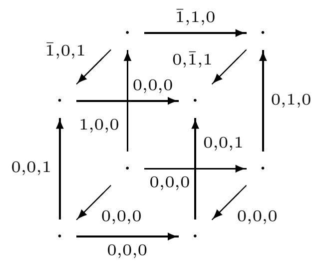
case (a)



case (b)



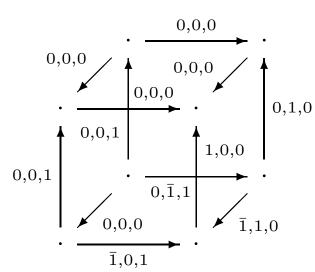
case (c)



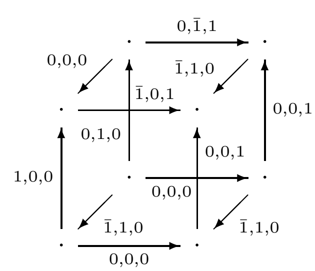
8.2.4. *(m, n) is interfering and upper-perfectible with m above n.* Then (M, N) is interfering and upper-perfectible with M above N. The set of cells $m \cup \tilde{n} = n \cup \tilde{m}$ is a horizontal strip, so there are no unexpected coincidences of modified columns. There are three interesting types of modified columns of S: (a) those which are positively modified columns of n, (b) those which are positively modified columns of m, and (c) those which are immediately to the right of negatively modified columns of m, and cause interference. Note that modified columns of S are not negatively modified columns of mn_{per} by maximality of S. Also note that all the modified columns of S in case (b) cause interference with n.

The edges of the cubes give vectors in the indices: $(-m)$ negatively modified columns of $m \cup mn_{\text{per}} \cup m_{\text{comp}}$, $(-n+m)$ negatively modified columns of n or positively modified columns of $m \cup m_{\text{comp}}$, and $(+n)$ positively modified columns of $n \cup n_{\text{comp}}$.

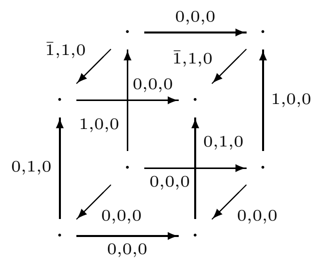
case (a)



case (b)



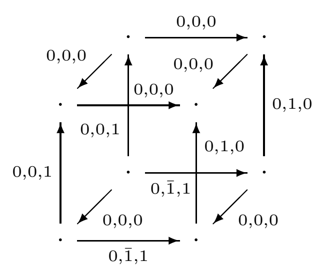
case (c)



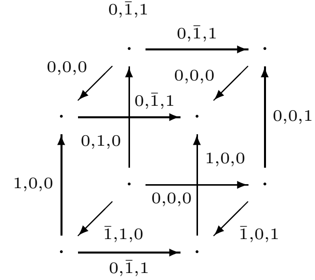
8.2.5. *(m,n) is interfering and lower-perfectible with m above n.* In this case (M,N) is interfering and lower-perfectible with M above N. We first observe that the positively modified columns of mn_{per} are not modified columns of S, for otherwise all the positively modified columns of n would be modified columns of S (and thus $N=\emptyset$). There are three interesting types of modified columns of S: (a) those which are positively modified columns of n, (b) those which are positively modified columns of m, and (c) those which are immediately to the right of negatively modified columns of m, and cause interference. Note that if m_{comp} is non-empty, the leftmost positively modified column of it will always interfere with the rightmost negatively modified column of $mn_{\text{per}} \subset \tilde{n}$ in the calculation of $\text{push}(S_m,\tilde{n})$.

The edges of the cubes give vectors in the indices: $(-m)$ negatively modified columns of $m \cup m_{\text{comp}}$, $(-n+m)$ negatively modified columns of $n \cup mn_{\text{per}}$ or positively modified columns of $m \cup m_{\text{comp}}$, and $(+n)$ positively modified columns of $n \cup mn_{\text{per}}$ (and also positively modified columns of the perfection arising from $\text{push}(S_m,\tilde{n})$).

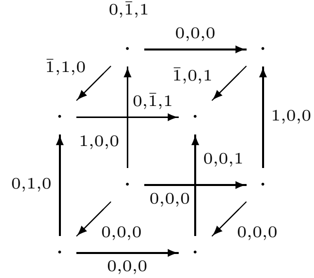
case (a)



case (b)



case (c)



II) m is a row move and n is a column move. We need to show that $\mathrm{ecs}(S,m)=\mathrm{cs}(\omega)=\mathrm{ecs}(S_n,\tilde{m})$ and $\mathrm{ers}(S,n)=\mathrm{rs}(\omega)=\mathrm{ers}(S_m,\tilde{n})$. We will prove the first equality, and the second one will follow from the same principles. We need to show that

$$\mathrm{per}_m(\mathrm{cs}(\lambda)+\Delta_{\mathrm{cs}}(S)+\Delta_{\mathrm{cs}}(m'))=\mathrm{per}_{\tilde{m}}(\mathrm{cs}(\kappa)+\Delta_{\mathrm{cs}}(S_n)+\Delta_{\mathrm{cs}}(\tilde{m}'))\,.$$

We have that $\mathrm{cs}(\lambda)=\mathrm{cs}(\kappa)$ (n is a column move) and $\Delta_{\mathrm{cs}}(S)=\Delta_{\mathrm{cs}}(S_n)$ (N is a column move and $\mathrm{cs}(\lambda)=\mathrm{cs}(\kappa)$). Since m and $\tilde{m}$ are row moves affecting the same columns (by definition of mixed equivalences) and S and S_n are strips with the same modified columns, we have that $\Delta_{\mathrm{cs}}(m')=\Delta_{\mathrm{cs}}(\tilde{m}')$. Therefore

$$\mathrm{cs}(\lambda)+\Delta_{\mathrm{cs}}(S)+\Delta_{\mathrm{cs}}(m')=\mathrm{cs}(\kappa)+\Delta_{\mathrm{cs}}(S_n)+\Delta_{\mathrm{cs}}(\tilde{m}')$$

The perfections of m and $\tilde{m}$ are the same (given $\Delta_{\mathrm{cs}}(m')=\Delta_{\mathrm{cs}}(\tilde{m}')$), and the equality follows.

III) m and n are column moves. The proof is basically the same as when m and n are row moves.

8.3. **Commuting cube (degenerate case $M = \emptyset$).** Suppose that $m, n, \tilde{m}$ and $\tilde{n}$ are non-empty and that $M = \emptyset$. Then one can check that we have one of the following two situations:

$$\begin{tikzcd}
& \mu \arrow[rr, "N"] \arrow[dl, "\emptyset"'] & & \rho \arrow[dl, "\emptyset"'] \\
\mu \arrow[rr, "N"] & & \rho & \\
& \lambda \arrow[uu, "S"'] \arrow[rr, "n"'] \arrow[dl, "m"] & & \kappa \arrow[uu, "S_n"'] \arrow[dl, "\tilde{m}"] \\
\nu \arrow[uu, "S\backslash m"] \arrow[rr, "\tilde{n}"'] & & \theta \arrow[uu, "\tilde{S}"'] &
\end{tikzcd} \tag{79}$$

or

$$\begin{tikzcd}
& \mu \arrow[rrrr, "N"] \arrow[dl, "\emptyset"'] & & & & \rho \arrow[dl, "\tilde{M}"'] \\
\mu \arrow[rr, "\tilde{N}"] & & \eta \arrow[rr, "x"] & & \sigma & \\
& \lambda \arrow[uu, "S"'] \arrow[rrrr, "n"'] \arrow[dl, "m"] & & & & \kappa \arrow[uu, "S_n"'] \arrow[dl, "\tilde{m}"] \\
\nu \arrow[uu, "S\backslash m"] \arrow[rr, "\tilde{n}"'] & & \theta \arrow[uu, "\hat{S}"'] \arrow[rr, "\emptyset"'] & & \theta \arrow[uu, "\tilde{S}"'] &
\end{tikzcd} \tag{80}$$

One obtains the commuting cube (79) except when (m, n) interferes and the perfection mn_{per} is made of strings that are translates of those of m, in which case one obtains the commuting cube (80). We consider the case that m and n are row moves as the column move case is similar. The exceptionsl situation can occur in two ways: either n is above m and the lower perfection exists, or m is above n and the upper perfection exists.

Suppose first that n is above m. Let $mn_{\mathrm{per}} = \bar{n}_I \cup \bar{n}_F$ where $\bar{n}_I$ are the strings of mn_{per} whose positively modified columns (resp. rows) are modified columns (resp. rows) of S. Suppose $n = n_I \cup n_F$ where n_I are the strings of n whose prolongation in mn_{per} is given by $\bar{n}_I$. Suppose (S, n) interferes with pushout perfection given by n_{comp} (if there is no interference then the situation is simpler). Then $(S_n, \tilde{m})$ also interferes; let $\bar{n}_{\mathrm{comp}}$ be the cells which define the pushout perfection. Finally, $(S \setminus m, \tilde{n})$ also interferes with pushout perfection given by $n_{\mathrm{comp}} \cup \bar{n}_{\mathrm{comp}}$. Then $N = n_I^+ \cup n_F^+ \cup n_{\mathrm{comp}}$, $\tilde{N} = n_F^+ \cup \bar{n}_F^+ \cup n_{\mathrm{comp}} \cup \bar{n}_{\mathrm{comp}}$, $\tilde{M} = \bar{n}_F^+ \cup \bar{n}_{\mathrm{comp}}$, and $x = n_I^+$ are such that $\tilde{M}N \equiv x\tilde{N}$ is an elementary equivalence. The last vertical face is then such that x is an augmentation move. Note that $\bar{n}_I$ may be empty in which case x is empty and we have an ordinary cube but with $\tilde{M} \neq \emptyset$.

Suppose m is above n. Let $n = n_I \cup n_F$ where n_I consists of the strings of n whose positively modified columns are positively modified columns of S. Let $mn_{\mathrm{per}} = \bar{n}_I \cup \bar{n}_F$ where $\bar{n}_I$ consists of the strings of mn_{per} that extend the strings of n_I above. Then $N = n_F^+ \cup n_{\mathrm{comp}}$ where n_{comp} is defined as in $\mathrm{push}(S, n)$ and $\tilde{N} = \bar{n}_F^+ \cup n_F^+ \cup m^+ \cup n_{\mathrm{comp}}$ since $S \setminus m$ and $\tilde{n}$ interfere and its completion is $m^+ \cup n_{\mathrm{comp}}$. With $\tilde{M} = \bar{n}_F \cup m^+$ and $x = \emptyset$ we are in the situation of the commuting cube (80).

8.4. **Commuting cube (degenerate case $m = \emptyset$).** Suppose that $m = \emptyset$ and that $\tilde{m}n = \tilde{n}\emptyset$ is an elementary equivalence. This situation can be seen as a degenerate

case of the $M = \emptyset$ case where we have the following commuting situation.

$$\begin{tikzcd}
 & \mu \arrow[rrr, "N"] \arrow[dl, "\emptyset"'] & & & \rho \arrow[dl, "\tilde{M}"'] \\
\mu \arrow[rr, "\tilde{N}"] & & \eta \arrow[r, "x"] & \sigma & \\
 & \lambda \arrow[uu, "S"] \arrow[rrr, "n"'] \arrow[dl, "\emptyset"] & & & \kappa \arrow[uu, "S_n"'] \arrow[dl, "\tilde{m}"] \\
\lambda \arrow[uu, "S"] \arrow[rr, "\tilde{n}"'] & & \theta \arrow[uu, "\hat{S}"'] \arrow[r, "\emptyset"'] & \theta \arrow[uu, "\tilde{S}"'] &
\end{tikzcd} \tag{81}$$

where vertical faces are either pushouts or augmentations moves, and where $\tilde{M}N \equiv x\tilde{N}$ is an elementary equivalence.

The case where x is non-empty will occur when $n = n_I \cup n_F$ and $\tilde{m} = \bar{n}_I \cup \bar{n}_F$ is such that the positively modified columns (resp. rows) of $\bar{n}_I$ are positively modified columns (resp. rows) of S. Then $N = n_I^+ \cup n_F^+$, $\tilde{M} = \bar{n}_F^+$, $\tilde{N} = n_F^+ \cup \bar{n}_F^+$ and $x = n_I^+$ are such that $\tilde{M}N \equiv x\tilde{N}$ is an elementary equivalence. The last vertical face is then such that x is an augmentation move.

8.5. **Commuting cube (degenerate case $\tilde{m} = \emptyset$).** This case is similar to the $m = \emptyset$ case. Another way to see this case is to consider that if we have $\tilde{n}m \equiv \emptyset n$ then we can use $\emptyset n \equiv n \emptyset$ (which leads trivially to a commuting cube) to fall back on the already treated $n = \emptyset$ case (which is equal to $m = \emptyset$ by symmetry).

9. Pullbacks

Given a strip $S = \mu/\lambda$ and a class of paths $[\mathbf{p}]$ in the k-shape poset from λ to ν, the pushout algorithm gives rise to a maximal strip $\tilde{S} = \eta/\nu$ and a unique class of paths $[\mathbf{q}]$ in the k-shape poset from μ to some η:

$$\begin{tikzcd}
\lambda \arrow[r, "{[\mathbf{p}]}"] \arrow[d, "S"'] & \nu \arrow[d, "\tilde{S}"] \\
\mu \arrow[r, "{[\mathbf{q}]}"] & \eta
\end{tikzcd} \tag{82}$$

Our goal is to show that this process is invertible when the strip S is reverse maximal. That is, given the maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$ in the k-shape poset from μ to η, we will describe a *pullback algorithm* that gives back the maximal strip $S = \mu/\lambda$ and the class of paths $[\mathbf{p}]$ in the k-shape poset from λ to ν. To indicate that we are in the pullback situation, the direction of the arrows will be reversed

$$\begin{tikzcd}
\lambda & \nu \arrow[l, "{[\mathbf{p}]}"'] \\
\mu \arrow[u, "S"] & \eta \arrow[l, "{[\mathbf{q}]}"'] \arrow[u, "\tilde{S}"']
\end{tikzcd} \tag{83}$$

The situation in the reverse case is quite similar to the situation we have encountered so far (which we will refer to as the *forward* case). We will establish a dictionary that allows to translate between the forward and reverse cases. Then only the main results will be stated.

10. Equivalences in the reverse case

If $m = \mu/\lambda$ is a move from λ then we say that m is a *move to* μ. We write $m\#\mu = \lambda$. We will use the same notation for the string decomposition $m = s_1 \cup \cdots \cup s_\ell$ of the forward case also in the reverse situation. That is, string s_1 is the leftmost and string s_ℓ is the rightmost. The following dictionary translates between the forward and reverse situations:

λ	$\longleftrightarrow$	μ
move m from λ	$\longleftrightarrow$	move m to μ
$\mu = m * \lambda$	$\longleftrightarrow$	$\lambda = m\#\mu$
leftmost (rightmost) string of m	$\longleftrightarrow$	rightmost (leftmost) string of m
continues below (resp. above)	$\longleftrightarrow$	continues below (resp. above)
column to the right (resp. left)	$\longleftrightarrow$	column to the left (resp. right)
row above (resp. below)	$\longleftrightarrow$	row below (resp. above)
shifting to the right (resp. up)	$\longleftrightarrow$	shifting to the left (resp. down)

Notation 156. For two sets of cells X and Y, let $\leftarrow_X (Y)$ (resp. $\downarrow_X (Y)$) denote the result of shifting to the left (resp. down), each row (resp. column) of Y by the number of cells of X in that row (resp. column).

10.1. **Reverse mixed elementary equivalence.** Let $\tilde{m}$ and $\tilde{M}$ be respectively a row move and a column move to γ. The contiguity of two moves is defined as in the forward case (that is, whether two disjoint strings can be joined to form one string).

Definition 157. A *reverse mixed elementary equivalence* is a relation of the form (42) satisfying (43) arising from a row move $\tilde{m}$ and column move $\tilde{M}$ to some $\gamma \in \Pi^k$, which has one of the following forms:

(1) $\tilde{m}$ and $\tilde{M}$ do not intersect and no cell of $\tilde{m}$ is contiguous to a cell of $\tilde{M}$. Then $m = \tilde{m}$ and $M = \tilde{M}$.
(2) $\tilde{m}$ and $\tilde{M}$ intersect and
   (a) $\tilde{m}$ continues above and below $\tilde{M}$. Then
$$m = \leftarrow_{\tilde{M}} (\tilde{m}) \qquad \text{and} \qquad M = \leftarrow_{\tilde{m}} (\tilde{M})$$
   (b) $\tilde{M}$ continues above and below $\tilde{m}$. Then
$$m = \downarrow_{\tilde{M}} (\tilde{m}) \qquad \text{and} \qquad M = \downarrow_{\tilde{m}} (\tilde{M}).$$

Proposition 158. *If $(\tilde{m}, \tilde{M})$ defines a reverse mixed elementary equivalence, then the prescribed sets of cells m and M are reverse moves such that $m \cup \tilde{M} = M \cup \tilde{m}$ (that is, there is a shape $\lambda = \tilde{M}\#(m\#\gamma)$ such that the diagram* (43) *commutes), and* (44) *holds.*

Mixed elementary equivalences and reverse mixed elementary equivalences are inverse operations in the following sense.

Proposition 159.

(1) *Suppose (m, M) is a (forward) mixed elementary equivalence. Then $(\tilde{m}, \tilde{M})$ is a reverse mixed elementary equivalence (determining (m, M)).*

(2) *Suppose $(\tilde{m}, \tilde{M})$ is a reverse mixed elementary equivalence. Then (m, M) is a mixed elementary equivalence (determining $(\tilde{m}, \tilde{M})$).*

Furthermore in both cases, the type – (1), (2)(a), (2)(b) – of the equivalence is preserved (see Definition 40).

10.2. **Reverse row elementary equivalence.** Let $\tilde{m}$ and $\tilde{M}$ be row moves to γ. We say that $(\tilde{m}, \tilde{M})$ is interfering if $\tilde{m}$ and $\tilde{M}$ do not intersect and $\gamma \setminus (\tilde{m} \cup \tilde{M})$ is not a k-shape (or to be more precise $\mathrm{cs}(\gamma \setminus (\tilde{m} \cup \tilde{M}))$ is not a partition). Let $\tilde{m} = s_1 \cup \cdots \cup s_r$ and $\tilde{M} = s'_1 \cup \cdots \cup s'_{r'}$ *interfere*. We immediately have

Lemma 160. *Suppose $(\tilde{m}, \tilde{M})$ is interfering and the top cell of $\tilde{m}$ is above the top cell of $\tilde{M}$. Then*

1. $c_{s'_1,u} = c^+_{s_r,d}$. *In particular, $\tilde{m}$ and $\tilde{M}$ are non-degenerate.*
2. *Every cell of m is above every cell of M.*
3. $\mathrm{cs}(\gamma)_{c_{s_r,d}} = \mathrm{cs}(\gamma)_{c_{s'_1,u}} + 1$.

Remark 161. Lemma 160 illustrates well how the forward-reverse dictionary is used. The condition for interference in the forward case is $c^-_{s_1,d} = c_{s'_{r'},u}$ and $\mathrm{cs}(\lambda)_{c_{s_1,d}} = \mathrm{cs}(\lambda)_{c_{s'_{r'}+1}} + 1$ which translates into $c^+_{s_r,d} = c_{s'_1,u}$ and $\mathrm{cs}(\gamma)_{c_{s_r,d}} = \mathrm{cs}(\gamma)_{c_{s'_1+1}} + 1$ in the reverse case.

Suppose $\tilde{m}$ is a move of rank r and length ℓ and $\tilde{M}$ is a move of rank r' and length ℓ', both to γ. Suppose also that $(\tilde{m}, \tilde{M})$ is interfering and that the top cell of $\tilde{m}$ is above the top cell of $\tilde{M}$. A *lower perfection* (resp. *upper perfection* is a k-shape of the form $\gamma \setminus (\tilde{m} \cup \tilde{M} \cup m_{\mathrm{per}})$ (resp. $\gamma \setminus (\tilde{m} \cup \tilde{M} \cup M_{\mathrm{per}})$) where m_{per} (resp. M_{per}) is a $(\gamma \setminus (\tilde{m} \cup \tilde{M}))$-removable skew shape such that $\tilde{m} \cup m_{\mathrm{per}}$ (resp. $\tilde{M} \cup M_{\mathrm{per}}$) is a row move to $\tilde{M} \# \lambda$ (resp. $\tilde{m} \# \lambda$) of rank r (resp. r') and length $\ell + \ell'$ and $\tilde{M} \cup m_{\mathrm{per}}$ (resp. $\tilde{m} \cup M_{\mathrm{per}}$) is a row move to $\tilde{m} \# \gamma$ (resp. $\tilde{M} \# \gamma$) of rank $r + r'$ and length ℓ' (resp. ℓ). If $(\tilde{m}, \tilde{M})$ is interfering then it is *lower* (resp. *upper*) *perfectible* if it admits a lower (resp. upper) perfection.

Definition 162. A *reverse row elementary equivalence* is a relation of the form (42) satisfying (43) arising from two row moves $\tilde{m}$ and $\tilde{M}$ to some k-shape γ, which has one of the following forms:

1. $\tilde{m}$ and $\tilde{M}$ do not intersect and $\tilde{m}$ and $\tilde{M}$ do not interfere. Then $m = \tilde{m}$ and $M = \tilde{M}$.
2. $(\tilde{m}, \tilde{M})$ is interfering (and say the top cell of $\tilde{m}$ is above the top cell of $\tilde{M}$) and is lower (resp. upper) perfectible by adding cells m_{per} (resp. M_{per}). Then $m = \tilde{m} \cup m_{\mathrm{per}}$ (resp. $m = \tilde{m} \cup M_{\mathrm{per}}$) and $M = \tilde{M} \cup m_{\mathrm{per}}$ (resp. $M = \tilde{M} \cup M_{\mathrm{per}}$).
3. $\tilde{m}$ and $\tilde{M}$ intersect and are matched above (resp. below). In this case $m = \tilde{m} \setminus (\tilde{m} \cap \tilde{M})$ and $M = \tilde{M} \setminus (\tilde{m} \cap \tilde{M})$.
4. $\tilde{m}$ and $\tilde{M}$ intersect and $\tilde{m}$ continues above and below $\tilde{M}$. In this case $m = \downarrow_{\tilde{m} \cap \tilde{M}} (\tilde{m})$ and $M = \downarrow_{\tilde{m} \cap \tilde{M}} (\tilde{M})$.
5. $\tilde{M} = \emptyset$ and there is a row move m_{per} to $\tilde{m} \# \gamma$ such that $\tilde{m} \cup m_{\mathrm{per}}$ is a row move to γ. Then $M = m_{\mathrm{per}}$ and $m = \tilde{m} \cup m_{\mathrm{per}}$.

In case (2),(4) and (5) the roles of $\tilde{m}$ and $\tilde{M}$ may be exchanged.

Proposition 163.

(1) *Suppose* (m, M) *defines a (forward) row elementary equivalence that produces the pair* $(\tilde{m}, \tilde{M})$*. Then* $(\tilde{m}, \tilde{M})$ *defines a reverse row elementary equivalence that produces* (m, M)*.*
(2) *Suppose* $(\tilde{m}, \tilde{M})$ *defines a reverse row elementary equivalence that produces the pair* (m, M)*. Then* (m, M) *defines a row elementary equivalence that produces* $(\tilde{m}, \tilde{M})$*.*

Furthermore we have:

- (m, M) *is in Case (1) if and only if* $(\tilde{m}, \tilde{M})$ *is in Case (1)*
- (m, M) *is in Case (2) or (5) if and only if* $(\tilde{m}, \tilde{M})$ *is in Case (3)*
- (m, M) *is in Case (3) if and only if* $(\tilde{m}, \tilde{M})$ *is in Case (2) or (5)*
- (m, M) *is in Case (4) if and only if* $(\tilde{m}, \tilde{M})$ *is in Case (4)*

Remark 164. According to Proposition 163, it would seem natural to join Cases (2) and (5) of forward and reverse row elementary equivalences under a single case. Indeed, these are the only two cases that need perfections and one can think of Case (5) as a degeneration of Case (2). However, due to the special nature of Case (5) (the presence of an empty move), we prefer not to merge the two cases.

10.3. **Reverse column elementary equivalence.** There is an obvious transpose analogue of reverse row elementary equivalences which we shall call *reverse column elementary equivalences*.

10.4. **Reverse diamond equivalences are generated by reverse elementary equivalences.** A reverse diamond equivalence is just a usual diamond equivalence $\tilde{M}m \equiv \tilde{m}M$ except that, instead of starting with (m, M) and producing $(\tilde{m}, \tilde{M})$, we start with $(\tilde{m}, \tilde{M})$ and produce (m, M). The next proposition follows immediately from the forward situation and Proposition 163.

Proposition 165. *The equivalence relations generated respectively by reverse diamond equivalences and by reverse elementary equivalences are identical.*

11. Reverse operations on strips

If $S = \mu/\lambda$ is a strip on λ then we say that S is a strip *inside* μ. To translate between the forward and reverse situations we add these elements to our dictionary:

strip $S = \mu/\lambda$ on λ	$\longleftrightarrow$	strip $S = \mu/\lambda$ inside μ
μ	$\longleftrightarrow$	λ
λ	$\longleftrightarrow$	μ
$\mathrm{bot}_c(\gamma)$	$\longleftrightarrow$	cell below $\mathrm{bot}_c(\gamma)$
k	$\longleftrightarrow$	$k+1$
lower (upper) augmentable corner	$\longleftrightarrow$	lower (upper) reverse augmentable corner
addable corner	$\longleftrightarrow$	removable corner
maximal strip (cover)	$\longleftrightarrow$	reverse maximal strip (cover)
$\Delta_{\mathrm{rs}}(S)$	$\longleftrightarrow$	$-\Delta_{\mathrm{rs}}(S)$
$\Delta_{\mathrm{rs}}(m)$	$\longleftrightarrow$	$-\Delta_{\mathrm{rs}}(m)$

Definition 166. Let $\mu \in \Pi$ be fixed. Let $\mathrm{Strip}^\mu \subset \Pi$ be the induced subgraph of $\nu \in \Pi$ such that μ/ν is a strip inside μ. If $\tilde{m}$ is a move such that $\lambda = \tilde{m}\#\nu$ in Strip^μ we shall say that $\tilde{m}$ is a reverse μ-augmentation move from the strip μ/ν to the strip μ/λ. A reverse augmentation of a strip $\tilde{S} = \mu/\lambda$ is a strip reachable from $\tilde{S}$ via a reverse μ-augmentation path. A strip $\tilde{S} = \mu/\lambda$ is reverse maximal if it admits no reverse μ-augmentation move.

Diagrammatically, a reverse augmentation move is such that the following diagram commutes for strips S and $\tilde{S}$ inside μ.

$$\begin{tikzcd}
\lambda & \nu \arrow[l, "\tilde{m}"'] \\
\mu \arrow[u, "S"] & \mu \arrow[l, "\emptyset"] \arrow[u, "\tilde{S}"']
\end{tikzcd}$$

These definitions depend on a fixed $\mu \in \Pi$, which shall usually be suppressed in the notation. Later we shall consider reverse augmentations of a given strip $\tilde{S}$, meaning reverse μ-augmentations where $\tilde{S} = \mu/\lambda$.

Proposition 167. *All reverse augmentation column moves of a strip $\tilde{S} = \mu/\lambda$ have rank 1.*

Let $\tilde{S} = \mu/\lambda$ be a strip and a be a removable corner of λ. We will call a a

(1) *lower reverse augmentable corner* of $\tilde{S}$ if removing a from λ adds a box to $\partial\lambda$ in a modified column c of $\tilde{S}$.
(2) *upper reverse augmentable corner* of $\tilde{S}$ if a does not lie below a box in $\tilde{S}$ and removing a from λ adds a box to $\partial\lambda$ in a modified row r of $\tilde{S}$.

A λ-removable string s of row-type (resp. column-type) can be reverse extended below (resp. above) if there is a λ-removable corner contiguous and below (resp. above) the lowest (resp. highest) cell of s.

Definition 168. A *reverse completion row move* is one in which all strings start in the same row. It is maximal if the first string cannot be reverse extended below. A *reverse quasi-completion column move* is a reverse column augmentation move from

a strip $\tilde{S}$ that contains no lower reverse augmentable corner. A *reverse completion column move* is a reverse quasi-completion move from a strip $\tilde{S}$ that contains no upper reverse augmentable corner below its unique (by Proposition 167) string. A reverse completion column move or a reverse quasi-completion column move is maximal if its string cannot be reverse extended above. A reverse completion move is a reverse completion row/column move.

Proposition 169. *Let $\tilde{S} = \mu/\lambda$ be a strip.*

(1) *$\tilde{S}$ has a unique maximal reverse augmentation $\tilde{S}' \in \mathrm{Strip}^\mu$.*
(2) *There is one equivalence class of paths in Strip^μ from $\tilde{S}$ to $\tilde{S}'$.*
(3) *The unique equivalence class of paths in Strip^μ from $\tilde{S}$ to $\tilde{S}'$ has a representative consisting entirely of maximal reverse completion moves.*

11.1. **Reverse maximal strips.**

Proposition 170. *A strip $\tilde{S}$ is reverse maximal if and only if it has no reverse augmentable corners.*

Proposition 171. *Suppose μ is a $(k+1)$-core and $\tilde{S} = \mu/\lambda$ is a reverse maximal strip. Then λ is a $(k+1)$-core.*

12. Pullback of strips and moves

Let $(\tilde{S}, \tilde{m})$ be a final pair where $\tilde{S} = \eta/\nu$ is a strip and $\tilde{m} = \eta/\mu$ is a nonempty row move.

We say that $(\tilde{S}, \tilde{m})$ is compatible if it is reasonable, not contiguous, (and normal if $\tilde{m}$ is a column move) and is either (1) non-interfering, or (2) is interfering but is also pullback-perfectible; all these notions are defined below.

For compatible pairs $(\tilde{S}, \tilde{m})$ we define a k-shape $\lambda \in \Pi$ (see Subsections 12.4 and 12.8 for case (1) and 12.5 and 12.9 for case (2)). This given, we define the pullback

$$\mathrm{pull}(\tilde{S}, \tilde{m}) = (S, m) = (\mu/\lambda, \nu/\lambda) \tag{84}$$

which produces an initial pair (S, m) where S is a strip and m is a (possibly empty) move. This is depicted by the following diagram.

$$\begin{tikzcd} \lambda & \nu \arrow[l, dotted, "m"'] \\ \mu \arrow[u, dotted, "S"] & \eta \arrow[l, "\tilde{m}"] \arrow[u, "\tilde{S}"'] \end{tikzcd}$$

If $\tilde{S}$ is a reverse maximal strip then $(\tilde{S}, \tilde{m})$ is compatible by Corollaries 176 and 180.

12.1. **Reasonableness.**

We say that the pair $(\tilde{S}, \tilde{m})$ is *reasonable* if for every string $s \subset \tilde{m}$, either $s \cap \tilde{S} = \emptyset$ or $s \subset \tilde{S}$.

Proposition 172. *Let $(\tilde{S}, \tilde{m})$ be a final pair with $\tilde{S}$ is a reverse maximal strip. Then $(\tilde{S}, \tilde{m})$ is reasonable.*

12.2. **Contiguity.**

We say that $(\tilde{S}, \tilde{m})$ is *contiguous* if there is a box $b \notin \partial\mu \cup \partial\nu$ which is present in $\partial(\mu \cap \nu)$. Call such a b an *appearing box.*

Proposition 173. *Let $(\tilde{S}, \tilde{m})$ be a final pair with $\tilde{S}$ a reverse maximal strip. Then $(\tilde{S}, \tilde{m})$ is non-contiguous.*

12.3. **Row-type pullback: interference.** Suppose that $(\tilde{S}, \tilde{m})$ is reasonable and non-contiguous with $\tilde{m}$ a row move. If $s \subset \tilde{m}$ is contained inside $\tilde{S}$, we say that $\tilde{S}$ *matches* s below if $c_{s,d}$ is a modified column of $\tilde{S}$. Otherwise we say that $\tilde{S}$ *continues below* s. Define $\tilde{m}'$ and $\tilde{m}'$ by

$$\tilde{m}' = \bigcup \{ \text{strings } s \subset \tilde{m} \mid s \text{ and } \tilde{S} \text{ are not matched below} \} \tag{85}$$

$$\tilde{m}^- = \downarrow_{\tilde{S}} \tilde{m}'. \tag{86}$$

We say that $(\tilde{S}, \tilde{m})$ is *non-interfering* if $\mathrm{cs}(\eta) - \Delta_{\mathrm{cs}}(\tilde{S}) - \Delta_{\mathrm{cs}}(\tilde{m}')$ is a partition and is *interfering* otherwise.

12.4. **Row-type pullback: non-interfering case.** Assume that $(\tilde{S}, \tilde{m})$ reasonable, non-contiguous, and non-interfering with $\tilde{m}$ a row move. Then we define $(\tilde{S}, \tilde{m})$ to be compatible, with $\lambda = \tilde{m}^- \# \nu$ and define $\mathrm{pull}(\tilde{S}, \tilde{m})$ by (84).

Proposition 174. *Let $(\tilde{S}, \tilde{m})$ be a reasonable, non-contiguous and non-interfering final pair. Then μ/λ is a strip.*

12.5. **Row-type pullback: interfering case.** Assume that $(\tilde{S}, \tilde{m})$ is reasonable, non-contiguous, and interfering with $\tilde{m}$ a row move. Say that $(\tilde{S}, \tilde{m})$ is *pullback-perfectible* if there is a set of cells $\tilde{m}_{\mathrm{comp}}$ inside $(\tilde{m}^-)\#\nu$ so that if $\lambda = ((\tilde{m}^-)\#\nu) \setminus \tilde{m}_{\mathrm{comp}}$ then μ/λ is a strip and ν/λ is a row move to ν with the same initial string as $\tilde{m}^-$.

Proposition 175. *Suppose $(\tilde{S}, \tilde{m})$ is a reasonable, non-contiguous, interfering final pair such that $\tilde{m}$ is a row move and $\tilde{S}$ is a reverse maximal strip. Then $(\tilde{S}, \tilde{m})$ is pullback-perfectible. Furthermore the strings of $\tilde{m}_{\mathrm{comp}}$ lie on exactly the same rows as the initial string of $\tilde{m}$.*

Corollary 176. *Suppose $(\tilde{S}, \tilde{m})$ is a final pair such that $\tilde{S}$ is a reverse maximal strip and $\tilde{m}$ is a row move. Then $(\tilde{S}, \tilde{m})$ is compatible.*

12.6. **Column-type pullback: normality.** Suppose that $(\tilde{S}, \tilde{m})$ is a reasonable final pair with $\tilde{m}$ a column move. If $s \subset \tilde{m}$ is contained inside $\tilde{S}$, we say that $\tilde{S}$ *matches* s *above* if $r_{s,u}$ is a modified row of $\tilde{S}$. Otherwise we say that $\tilde{S}$ *continues above* s.

Let $s \subset \tilde{m}$ be the final string of the move $\tilde{m}$. We say that $(\tilde{S}, \tilde{m})$ is *normal* if it is reasonable, and, in the case that s is continued above, (a) none of the modified rows of $\tilde{S}$ contains boxes of s and (b) the negatively modified row of s is not a modified row of S.

Proposition 177. *Let $\tilde{S}$ be a reverse maximal strip and $\tilde{m}$ a column move. Then $(\tilde{S}, \tilde{m})$ is normal and non-contiguous.*

12.7. **Column-type pullback: interference.** Define $\tilde{m}'$ and $\tilde{m}^-$ by

$$\tilde{m}' = \bigcup \{ \text{strings } s \subset \tilde{m} \mid s \text{ and } \tilde{S} \text{ are not matched above} \} \tag{87}$$

$$\tilde{m}^- = \leftarrow_{\tilde{S}} \tilde{m}'. \tag{88}$$

If $\tilde{m}' \neq \emptyset$ we say that $(\tilde{S}, \tilde{m})$ is *non-interfering* if $\mathrm{rs}(\eta) - \Delta_{\mathrm{rs}}(\tilde{S}) - \Delta(\tilde{m}')$ is a partition and *interfering* otherwise. If $\tilde{m}' = \emptyset$ we say that $(\tilde{S}, \tilde{m})$ is non-interfering if $\mathrm{rs}(\mu)/\mathrm{rs}(\nu)$ is a horizontal strip and interfering otherwise (observe that $\mathrm{rs}(\eta) - \Delta_{\mathrm{rs}}(\tilde{S}) - \Delta(\tilde{m}') = \mathrm{rs}(\eta) - \Delta_{\mathrm{rs}}(\tilde{S}) = \mathrm{rs}(\nu)$ is always a partition in that case). The latter case is referred to as *special interference*.

12.8. **Column-type pullback: non-interfering case.** Assume that $(\tilde{S}, \tilde{m})$ is normal, non-contiguous and non-interfering with $\tilde{m}$ a column move. In this case we declare $(\tilde{S}, \tilde{m})$ to be compatible. $\tilde{m}^-$ is a move to ν and we define $\lambda = \tilde{m}^- \# \nu$. The pullback is defined by (84).

Proposition 178. *Suppose $(\tilde{S}, \tilde{m})$ is normal, non-contiguous and non-interfering. Then μ/λ is a strip.*

12.9. **Column-type pullback: interfering case.** Assume that $(\tilde{S}, \tilde{m})$ is normal, non-contiguous and interfering with $\tilde{m}$ a column move. Say that $(\tilde{S}, \tilde{m})$ is *pullback-perfectible* if there is a set of cells $\tilde{m}_{\mathrm{comp}}$ inside $(\tilde{m}^-)\#\nu$ so that if $\lambda = ((\tilde{m}^-)\#\nu) \setminus \tilde{m}_{\mathrm{comp}}$ then μ/λ is a strip and ν/λ is a row move to ν with the same initial string as $\tilde{m}^-$. In the case that $(\tilde{S}, \tilde{m})$ is pullback-perfectible, we declare that $(\tilde{S}, \tilde{m})$ is compatible and use the above λ to define the pullback via (84).

Proposition 179. *Suppose $(\tilde{S}, \tilde{m})$ is reasonable, normal, non-contiguous and interfering with $\tilde{m}$ a column move and $\tilde{S}$ a reverse maximal strip. Then $(\tilde{S}, \tilde{m})$ is pullback-perfectible. Furthermore $\tilde{m}_{\mathrm{comp}}$ consists of a single string that lies on the same columns as the initial string of $\tilde{m}$.*

Corollary 180. *Suppose $(\tilde{S}, \tilde{m})$ is a final pair with $\tilde{S}$ a reverse maximal strip and $\tilde{m}$ a column move. Then $(\tilde{S}, \tilde{m})$ is compatible.*

13. Pullbacks sequences are all equivalent

Given a strip $\tilde{S} = \eta/\nu$ and a path $\mathbf{q}$ in the k-shape poset from μ to η, one can do a sequence of pullbacks and reverse augmentations to obtain a reverse maximal strip $S = \mu/\lambda$ and a path $\mathbf{p}$ in the k-shape poset from λ to ν:

$$\begin{tikzcd} \lambda & \nu \arrow[l, "\mathbf{p}"'] \\ \mu \arrow[u, "S"] & \eta \arrow[l, "\mathbf{q}"] \arrow[u, "\tilde{S}"'] \end{tikzcd} \tag{89}$$

Such a process, which we will call a *pullback sequence*, can always be done since we have seen that a reverse maximal strip is compatible with any move. As in the forward case, it does not matter which pullout sequence is used since they give rise to equivalent paths (and therefore to a unique reverse maximal strip S).

Proposition 181. *Let $\tilde{S} = \eta/\nu$ be strip and $\mathbf{q}$ a path in the k-shape poset from μ to η, and suppose that a given pullback sequence gives rise to a reverse maximal strip $S = \mu/\lambda$ and a path $\mathbf{p}$ in the k-shape poset from λ to ν. Then any other given pullback sequence gives rise to the reverse maximal strip $S = \mu/\lambda$ and a path $\tilde{\mathbf{p}}$ equivalent to $\mathbf{p}$.*

14. Pullbacks of equivalent paths are equivalent

The next proposition tells us that the pullbacks of equivalent paths produce equivalent paths.

Proposition 182. *Let $\tilde{S} = \eta/\nu$ be a strip and and let $\mathbf{q}$ and $\mathbf{q}'$ be equivalent paths in the k-shape poset from μ to η. If the pullback sequence associated to $\tilde{S}$ and $\mathbf{q}$ gives rise to a reverse maximal strip $S = \mu/\lambda$ and a path $\mathbf{p}$ in the k-shape poset*

from λ to ν, then the pullback sequence associated to $\tilde{S}$ and $\mathbf{q}'$ gives rise to the same reverse maximal strip $S = \mu/\lambda$ and a path $\mathbf{p}'$ equivalent to $\mathbf{p}$.

Propositions 182 and Proposition 181 provide an algorithm, which we will call the *pullback algorithm*, that, given a strip $\tilde{S} = \eta/\nu$ and a class of paths $[\mathbf{q}]$ in the k-shape poset from μ to η, gives rise to a reverse maximal strip $S = \mu/\lambda$ and a unique class of paths $[\mathbf{p}]$ in the k-shape poset from λ to ν:

$$\begin{tikzcd} \lambda & \nu \arrow[l, "{[\mathbf{p}]}"'] \\ \mu \arrow[u, "S"] & \eta \arrow[l, "{[\mathbf{q}]}"'] \arrow[u, "\tilde{S}"'] \end{tikzcd} \tag{90}$$

15. Pullbacks are inverse to pushouts

Proposition 183.

(1) *Let (S, m) be a compatible initial pair with* $\mathrm{push}(S, m) = (\tilde{S}, \tilde{m})$. *If $\tilde{m}$ is not empty then $(\tilde{S}, \tilde{m})$ is a compatible final pair such that* $\mathrm{pull}(\tilde{S}, \tilde{m}) = (S, m)$. *If $\tilde{m}$ is empty then m is a reverse augmentation move on the strip $\tilde{S}$.*
(2) *If $\tilde{m}$ is an augmentation move on the strip S such that $\tilde{m} * S = \tilde{S}$, then $(\tilde{S}, \tilde{m})$ is a compatible final pair such that* $\mathrm{pull}(\tilde{S}, \tilde{m}) = (S, \emptyset)$.
(3) *Let $(\tilde{S}, \tilde{m})$ be a compatible final pair with* $\mathrm{pull}(\tilde{S}, \tilde{m}) = (S, m)$. *If m is not empty then (S, m) is a compatible initial pair such that* $\mathrm{push}(S, m) = (\tilde{S}, \tilde{m})$. *If m is empty then $\tilde{m}$ is an augmentation move on the strip S.*
(4) *If m is a reverse augmentation move on the strip $\tilde{S}$ such that $m \# \tilde{S} = S$, then (S, m) is a compatible initial pair such that* $\mathrm{push}(S, m) = (\tilde{S}, \emptyset)$.

Proof. The non-empty cases follow from the alternative descriptions of pushouts and its analogue for pullbacks via expected row and column shape. The empty cases are immediate. □

We now prove Theorem 75. As already mentioned after the statement of Theorem 75, it suffices to prove the case where S and T are single strips. That is, we need to show that given a reverse maximal strip $S = \mu/\lambda$ and a class of paths $[\mathbf{p}]$ from λ to ν, the pushout algorithm gives rise to a maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$ from μ to a η:

$$\begin{tikzcd} \lambda \arrow[r, "{[\mathbf{p}]}"] \arrow[d, "S"'] & \nu \arrow[d, "\tilde{S}"] \\ \mu \arrow[r, "{[\mathbf{q}]}"] & \eta \end{tikzcd} \tag{91}$$

if and only if given the maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$ from μ to η, the pullback algorithm gives rise to the reverse maximal strip $S = \mu/\lambda$ and the class of paths $[\mathbf{p}]$ from λ to ν:

$$\begin{tikzcd} \lambda & \nu \arrow[l, "{[\mathbf{p}]}"'] \\ \mu \arrow[u, "S"] & \eta \arrow[l, "{[\mathbf{q}]}"'] \arrow[u, "\tilde{S}"'] \end{tikzcd} \tag{92}$$

Suppose we are given a reverse maximal strip $S = \mu/\lambda$ and a class of paths $[\mathbf{p}]$, and suppose that the pushout algorithm leads to the maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$. As we have seen, this implies that any pushout sequence leads to the maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$. By Proposition 183, every pushout sequence can be reverted to give a pullback sequence from the maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$. This ensures that there is at least one pullback sequence from the maximal strip $\tilde{S} = \eta/\nu$ and the class of paths $[\mathbf{q}]$ that leads to the reverse maximal strip $S = \mu/\lambda$ and the class of paths $[\mathbf{p}]$. As we have seen, this implies that the pullback algorithm always leads to the reverse maximal strip $S = \mu/\lambda$ and the class of paths $[\mathbf{p}]$. Therefore the pullback of a pushout gives back the initial pair. We can prove that the pushout of a pullback gives back the final pair in a similar way. Note that for the bijection to work, we need S to be reverse maximal and $\tilde{S}$ to be maximal. This is because the pushout algorithm produces a maximal strip while the pullback algorithm yields a reverse maximal strip.

Appendix A. Tables of branching polynomials

We list here all the branching polynomials $\tilde{b}^{(k)}_{\mu\lambda}(t)$ for partitions of degree up to 6.

Degree 2:

$b^{(2)}_{\mu\lambda}$	1^2	2
1^2	1	t

Degree 3:

$b^{(2)}_{\mu\lambda}$	1^3	21
1^3	1	t^2

$b^{(3)}_{\mu\lambda}$	1^3	21	3
1^3	1	t	
21		1	t

Degree 4:

$b^{(2)}_{\mu\lambda}$	1^4	21^2	2^2
1^4	1	t^2+t^3	t^4

$b^{(3)}_{\mu\lambda}$	1^4	21^2	2^2	31
1^4	1		t^2	
21^2		1		
2^2			1	t

$b^{(4)}_{\mu\lambda}$	1^4	21^2	2^2	31	4
1^4	1	t			
21^2		1		t	
2^2			1		
31				1	t

Degree 5:

$b^{(2)}_{\mu\lambda}$	1^5	21^3	2^21
1^5	1	t^3+t^4	t^6

$b^{(3)}_{\mu\lambda}$	1^5	21^3	2^21	31^2	32
1^5	1	t^2	t^3		
21^3		1	t	t^2	
2^21			1	t	t^2

$b^{(4)}_{\mu\lambda}$	1^5	21^3	2^21	31^2	32	41
1^5	1		t^2			
21^3		1				
2^21			1		t	
31^2				1		
32					1	t

$b^{(5)}_{\mu\lambda}$	1^5	21^3	2^21	31^2	32	41	5
1^5	1	t					
21^3		1		t			
2^21			1				
31^2				1		t	
32					1		
41						1	t

Degree 6:

$b^{(2)}_{\mu\lambda}$	1^6	21^4	2^21^2	2^3
1^6	1	$t^3+t^4+t^5$	$t^6+t^7+t^8$	t^9

$b^{(3)}_{\mu\lambda}$	1^6	21^4	2^21^2	2^3	31^3	321	3^2
1^6	1	t^2	t^4				
21^4		1		t^2	t^2		
2^21^2			1		t	t^2	
2^3				1		t^2	t^3

$b^{(4)}_{\mu\lambda}$	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42
1^6	1			t^3					
21^4		1	t						
2^21^2			1				t^2		
2^3				1		t			
31^3					1				
321						1		t	
3^3							1		t

$b^{(5)}_{\mu\lambda}$	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42	51
1^6	1		t^2							
21^4		1								
2^21^2			1			t				
2^3				1						
31^3					1					
321						1			t	
3^3							1			
41^2								1		
42									1	t

$b^{(6)}_{\mu\lambda}$	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42	51	6
1^6	1	t									
21^4		1			t						
2^21^2			1								
2^3				1							
31^3					1			t			
321						1					
3^3							1				
41^2								1		t	
42									1		
51										1	t

References

[1] S. Assaf and S. Billey, private communication.

[2] J. Blasiak, Cyclage, catabolism, and the affine Hecke algebra, preprint arXiv:1001.1569.

[3] L.-C. Chen, Ph. D. Thesis, U. C. Berkeley, 2010.

[4] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Am. Math. Soc. **14** (2001), 941–1006.

[5] T. Lam, *Affine Stanley symmetric functions*, Amer. J. Math. **128** (2006), no. 6, 1553–1586.

[6] T. Lam, *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21** (2008), no. 1, 259–281.

[7] T. Lam, *Affine Schubert classes, Schur positivity, and combinatorial Hopf algebras* preprint, 2009, `arXiv:0906.0385`.

[8] T. Lam, L. Lapointe, J. Morse, and M. Shimozono, *Affine insertion and Pieri rules for the affine Grassmannian*, to appear in Memoirs of the AMS.

[9] L. Lapointe, A. Lascoux, and J. Morse, *Tableau atoms and a new Macdonald positivity conjecture*, Duke Math. J. **116** (2003), no. 1, 103–146.

[10] L. Lapointe and J. Morse, *Schur function analogs for a filtration of the symmetric function space*, J. Combin. Theory Ser. A **101** (2003), no. 2, 191–224.

[11] L. Lapointe and J. Morse, *Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur expansions*, J. Comb. Th. A **112** (2005), 44–81.

[12] L. Lapointe and J. Morse, *A k-tableau characterization of k-Schur functions*, Adv. Math. **213** (2007), no. 1, 183–204.

[13] L. Lapointe and J. Morse, *Quantum cohomology and the k-Schur basis*, Trans. Amer. Math. Soc. **360** (2008), 2021–2040.

[14] L. Lapointe and M.E. Pinto, in preparation.

[15] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, Clarendon Press, Oxford, 1995.

Department of Mathematics, University of Michigan, Ann Arbor MI 48109 USA.
E-mail address: `tfylam@umich.edu`

Instituto de Matemática Y Física, Universidad de Talca, Casilla 747, Talca, Chile.
E-mail address: `lapointe@inst-mat.utalca.cl`

Department of Mathematics, Drexel University, Philadelphia, PA 19104 USA.
E-mail address: `morsej@math.drexel.edu`

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061 USA.
E-mail address: `mshimo@vt.edu`